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Finite-action solutions of Yang-Mills equations on de Sitter dS_4 and anti-de Sitter AdS_4 spaces

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Abstract: We consider pure SU(2) Yang-Mills theory on four-dimensional de Sitter dS_4 and anti-de Sitter AdS_4 spaces and construct various solutions to the Yang-Mills equations. On de Sitter space we reduce the Yang-Mills equations via an SU(2)-equivariant ansatz to Newtonian mechanics of a particle moving in $\mathbb{R}^3$ under the influence of a quartic potential. Then we describe magnetic and electric-magnetic solutions, both Abelian and non-Abelian, all having finite energy and finite action. A similar reduction on anti-de Sitter space also yields Yang-Mills solutions with finite energy and action. We propose a lower bound for the action on both backgrounds. Employing another metric on AdS_4, the SU(2) Yang-Mills equations are reduced to an analytic continuation of the above particle mechanics from $\mathbb{R}^3$ to $\mathbb{R}^{2,1}$. We discuss analytical solutions to these equations, which produce infinite-action configurations. After a Euclidean continuation of dS_4 and AdS_4 we also present self-dual (instanton-type) Yang-Mills solutions on these backgrounds.

Keywords: Solitons Monopoles and Instantons, Differential and Algebraic Geometry, Classical Theories of Gravity, Confinement

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Contents

1 Introduction

Magnetic monopoles [1–3] and vortices [4] are playing an important role in the nonperturbative physics of 3+1 dimensional Yang-Mills-Higgs theory [5–7]. However, in pure gauge theory without any scalar fields there are no vortices or non-Abelian monopoles on Minkowski space $\mathbb{R}^{3,1}$. Yet, our universe appears to be asymptotically de Sitter (not Minkowski) at very early and very late times. This provides strong motivation for searching finite-action solutions in pure Yang-Mills theory on de Sitter space dS_4. Finding Yang-Mills solutions on anti-de Sitter space AdS_4 is also reasonable from the viewpoint of string-theory applications and from the AdS/CFT perspective. The construction of such solutions, both Abelian and non-Abelian, is the goal of our paper.[1] Some steps in this direction have been made in [9].

[1] We consider the spacetime background as non-dynamical, i.e. we ignore the backreaction on it. The coupled system is governed by the Einstein-Yang-Mills equations (for numerical solutions, see e.g. the review [8] and references therein). However, in such a more general setup it is practically impossible to obtain analytic non-Abelian solutions.

In this paper, we present a construction of smooth Abelian and non-Abelian solutions with both finite energy and action in pure Yang-Mills theory on de Sitter space dS_4 and anti-de Sitter space AdS_4. Other types of Yang-Mills solutions on anti-de Sitter space, also described in this paper, have infinite energy and action. We also write down instantons and quasi-instantons in de Sitter dS_4 and anti-de Sitter AdS_4 spaces. We postpone the issue of boundary conditions and study classical solutions for any kind of boundary condition.

The paper is organized as follows. Section 2 provides a description of de Sitter space, which is used in section 3 to explicitly construct Yang-Mills solutions on dS_4 and to compute their energy and action. Instantons on the Euclideanized background are the topic of section 4. The story is repeated for anti-de Sitter space AdS_4 in sections 5, 6 and 7. We conclude in section 8. Four appendices list the various metrics used in the paper, detail metrics on the spatial slices S^3 and AdS_3, and present explicit expressions for our Yang-Mills solutions in the fundamental and in the adjoint SU(2) representation on dS_4 in various coordinates.

2 Description of de Sitter space dS_4

Closed-slicing coordinates. Four-dimensional de Sitter space can be embedded into five-dimensional Minkowski space $\mathbb{R}^{4,1}$ as the one-sheeted hyperboloid

$$\delta_{ij}y^iy^j - (y^5)^2 = R^2 \qquad \text{where} \quad i,j=1,\dots,4\,. \tag{2.1}$$

Topologically, de Sitter space dS_4 is $\mathbb{R}\times S^3$, and one can introduce global coordinates (τ,χ,θ,ϕ) adapted to this topology by setting (see e.g. [10])

$$y^i = R\,\omega^i\cosh\tau\,,\quad y^5 = R\,\sinh\tau \qquad \text{with} \quad \tau\in\mathbb{R} \quad \text{and} \quad \delta_{ij}\,\omega^i\omega^j=1 \tag{2.2}$$

for $\omega^i = \omega^i(\chi,\theta,\phi)$ embedding S^3 into $\mathbb{R}^{4,0}$. A dimensionful time coordinate may be introduced as $\tilde\tau = R\,\tau$. The flat metric on $\mathbb{R}^{4,1}$ induces a metric on dS_4,

$$\mathrm{d}s^2 = R^2\left(-\mathrm{d}\tau^2 + \cosh^2\!\tau\,\mathrm{d}\Omega_3^2\right) \tag{2.3}$$

with $\mathrm{d}\Omega_3^2$ being the metric on the unit sphere $S^3\cong \mathrm{SU}(2)$.

On this unit S^3 we introduce an orthonormal basis $\{e^a\}$, $a=1,2,3$, of left-invariant one-forms satisfying

$$\mathrm{d}e^a + \varepsilon^a_{bc}\,e^b\wedge e^c = 0\,. \tag{2.4}$$

For an embedding $\{\omega^i\}$, the one-forms $\{e^a\}$ can be constructed via

$$e^a = -\eta^a_{ij}\,\omega^i\mathrm{d}\omega^j\,, \qquad \text{with} \quad \eta^a_{ij} = \begin{cases} \varepsilon^a_{ij} & \text{for} \quad i,j=1,2,3 \\ +\delta^a_i & \text{for} \quad j=4 \\ -\delta^a_j & \text{for} \quad i=4 \\ 0 & \text{for} \quad i=j=4 \end{cases} \tag{2.5}$$

denoting the self-dual 't Hooft symbols. The metric on S^3 is then obtained as

$$\mathrm{d}\Omega_3^2 = \left(e^1\right)^2 + \left(e^2\right)^2 + \left(e^3\right)^2. \tag{2.6}$$

In appendix B we explicitly present two prominent such embeddings and the corresponding one-forms and metric.

Conformal coordinates. One can rewrite the metric (2.3) on dS_4 in conformal coordinates (t, χ, θ, ϕ) by the time reparametrization [10]

$$t = \arctan(\sinh\tau) = 2\arctan\left(\tanh\frac{\tau}{2}\right) \qquad \Longleftrightarrow \qquad \frac{\mathrm{d}\tau}{\mathrm{d}t} = \cosh\tau = \frac{1}{\cos t}\,, \tag{2.7}$$

in which $\tau \in (-\infty, \infty)$ corresponds to $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$. The metric (2.3) in these coordinates reads

$$\mathrm{d}s^2 = \frac{R^2}{\cos^2 t}\left(-\mathrm{d}t^2 + \delta_{ab}e^a e^b\right) = \frac{R^2}{\cos^2 t}\,\mathrm{d}s^2_{\mathrm{cyl}}\,, \tag{2.8}$$

where

$$\mathrm{d}s^2_{\mathrm{cyl}} = -\mathrm{d}t^2 + \delta_{ab}e^a e^b \tag{2.9}$$

is the standard metric on the Lorentzian cylinder $\mathbb{R}\times S^3$. Hence, four-dimensional de Sitter space is conformally equivalent to the finite cylinder $\mathcal{I}\times S^3$ with the metric (2.9), where $\mathcal{I}$ is the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ parametrized by t.

Static coordinates. The sphere S^3 can be glued from a pair (S^3_+, S^3_-) of 3-balls and a 2-sphere S^2,

$$S^3 = S^3_+ \cup S^2 \cup S^3_-\,, \tag{2.10}$$

where S^3_+ is an 'upper hemisphere', S^3_- is the 'lower hemisphere', and the gluing surface[2] is the equatorial 2-sphere S^2. On any 'half' $\mathbb{R}\times S^3_\pm \cong \mathbb{R}^4$ of dS_4 one may introduce static coordinates $(\sigma, \rho, \theta, \phi)$ by taking

$$y^a = R\,\rho\,\lambda^a\,,\;\; y^4 = R\sqrt{1-\rho^2}\,\cosh\sigma\,,\;\; y^5 = R\sqrt{1-\rho^2}\,\sinh\sigma \quad \text{with} \quad \sigma\in\mathbb{R}\,,\;\; \rho\in[0,1) \tag{2.11}$$

and $\delta_{ab}\lambda^a\lambda^b = 1$ for

$$\lambda^1 = \sin\theta\,\sin\phi\,, \quad \lambda^2 = \sin\theta\,\cos\phi\,, \quad \lambda^3 = \cos\theta\,. \tag{2.12}$$

In this case, the induced metric on dS_4 comes out as

$$\mathrm{d}s^2 = R^2\left(-(1-\rho^2)\mathrm{d}\sigma^2 + \frac{\mathrm{d}\rho^2}{1-\rho^2} + \rho^2\mathrm{d}\Omega_2^2\right) \qquad \text{with} \quad \mathrm{d}\Omega_2^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\,. \tag{2.13}$$

Dimensionful time and radial coordinates are $\mathrm{t} = R\,\sigma$ and $\mathrm{r} = R\,\rho$. Obviously, the metric (2.13) has a cosmological event horizon at $\rho{=}1$ (or $\mathrm{r}{=}R$) and, therefore, static coordinates cover only half of dS_4. In its range it is convenient to introduce a coordinate α instead of ρ via

$$\rho = \sin\alpha \qquad \Rightarrow \qquad \sqrt{1-\rho^2} = \cos\alpha\,, \tag{2.14}$$

that will be used in later calculations.

[2]See appendix B for more details. In particular, for the metric (B.2) we may take the angle $0 \le \chi < \frac{\pi}{2}$ for S^3_+, $\frac{\pi}{2} < \chi \le \pi$ for S^3_- and $\chi = \frac{\pi}{2}$ for the equatorial S^2.

3 Yang-Mills configurations on dS_4

In Minkowski space $\mathbb{R}^{3,1}$ smooth vortex or monopole solutions of gauge theory can be constructed only in the presence of Higgs fields. Both types of solutions have finite energy per length or energy, defined via integrals over $\mathbb{R}^{2,0} \subset \mathbb{R}^{3,1}$ or $\mathbb{R}^{3,0} \subset \mathbb{R}^{3,1}$, respectively. There are no finite-energy solutions of such kind in pure Yang-Mills theory in Minkowski space. Here, we will show that finite-energy solutions in gauge theory without scalar fields do exist on de Sitter space dS_4. Furthermore, they also have finite action, contrary to monopoles or vortices in $\mathbb{R}^{3,1}$.

Conformal invariance. Since in four dimensions the Yang-Mills equations are conformally invariant, their solutions on de Sitter space can be obtained by solving the equations on $\mathcal{I} \times S^3$ with the cylindrical metric (2.9). Therefore, we will consider a rank-N Hermitian vector bundle over the cylinder $\mathcal{I} \times S^3$ with a gauge potential $\mathcal{A}$ and the gauge field $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ taking values in the Lie algebra $su(N)$. The conformal boundary of dS_4 consists of the two 3-spheres at $t = \pm\frac{\pi}{2}$ or, equivalently, at $\tau = \pm\infty$. On a manifold M with boundary ∂M, gauge transformations are naturally restricted to tend to the identity when approaching ∂M (see e.g. [11]). This corresponds to a framing of the gauge bundle over the boundary. For our case, it means allowing only gauge-group elements $g(\cdot)$ subject to

$$g(\partial M) = \mathrm{Id} \qquad \text{on} \quad \partial M = S^3_{t=+\frac{\pi}{2}} \cup S^3_{t=-\frac{\pi}{2}} \,. \tag{3.1}$$

Reduction to matrix equations. In order to obtain explicit solutions we use the SU(2)-equivariant ansatz (cf. [12–14])

$$\mathcal{A} = X_a(t)\, e^a \tag{3.2}$$

for the $su(N)$-valued gauge potential $\mathcal{A}$ in the temporal gauge $\mathcal{A}_0 \equiv \mathcal{A}_t = 0 = \mathcal{A}_\tau$. Here, $X_a(t)$ are three $su(N)$-valued functions depending only on $t \in \mathcal{I}$, and e^a are the basis one-forms on S^3 satisfying (2.4). The corresponding gauge field reads

$$\mathcal{F} = \mathcal{F}_{0a}\, e^0 \wedge e^a + \frac{1}{2}\mathcal{F}_{bc}\, e^b \wedge e^c = \dot{X}_a\, e^0 \wedge e^a + \frac{1}{2}\big(-2\varepsilon^a_{bc} X_a + [X_b, X_c]\big) e^b \wedge e^c \,, \tag{3.3}$$

where $\dot{X}_a := \mathrm{d}X_a/\mathrm{d}t$ and $e^0 := \mathrm{d}t$. It is not difficult to show (see e.g. [14]) that the Yang-Mills equations on $\mathcal{I} \times S^3$ after substituting (3.2) and (3.3) reduce to the ordinary matrix differential equations

$$\ddot{X}_a = -4X_a + 3\varepsilon_{abc}\,[X_b, X_c] - \big[X_b, [X_a, X_b]\big] \qquad \text{and} \qquad [X_a, \dot{X}_a] = 0 \,. \tag{3.4}$$

Reduction to a Newtonian particle on $\mathbb{R}^3$. It is very natural to restrict the matrices X_a to an $su(2)$ subalgebra. To this end, we embed the spin-j representation of SU(2) into the fundamental of SU(N) with $N = 2j{+}1$. The three SU(2) generators I_a then obey

$$[I_b, I_c] = 2\,\varepsilon^a_{bc} I_a \quad \text{and} \quad \mathrm{tr}(I_a I_b) = -4\, C(j)\, \delta_{ab} \quad \text{for} \quad C(j) = \frac{1}{3}\, j(j{+}1)(2j{+}1) \,, \tag{3.5}$$

where $C(j)$ is the second-order Dynkin index of the spin-j representation. The simplest choice for X_a then is[3]

$$X_1 = \Psi_1 I_1\,, \quad X_2 = \Psi_2 I_2 \quad \text{and} \quad X_3 = \Psi_3 I_3\,, \tag{3.6}$$

where Ψ_a are real functions of $t \in \mathcal{I}$.

Due to the equivalence of Yang-Mills theory on dS_4 with metric (2.8) to the theory on $\mathcal{I} \times S^3$ with metric (2.9), we obtain the Lagrangian density

$$\begin{aligned} \mathcal{L} &= \frac{1}{8}\,\mathrm{tr}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} = -\frac{1}{4}\,\mathrm{tr}\mathcal{F}_{0a}\mathcal{F}_{0a} + \frac{1}{8}\,\mathrm{tr}\mathcal{F}_{ab}\mathcal{F}_{ab} \\ &= 4\,C(j)\Big\{\frac{1}{4}\dot{\Psi}_a\dot{\Psi}_a - (\Psi_1 - \Psi_2\Psi_3)^2 - (\Psi_2 - \Psi_3\Psi_1)^2 - (\Psi_3 - \Psi_1\Psi_2)^2\Big\}\,. \end{aligned} \tag{3.7}$$

The S^3 has disappeared, and we are left with at a Lagrangian on $\mathcal{I}$. Interpreting the real functions $\Psi_a(t)$ as coordinates of a particle on $\mathbb{R}^3$, this Lagrangian describes its Newtonian dynamics in a finite time interval, with kinetic energy T and quartic potential energy V,[4]

$$T = \frac{1}{2}\,\dot{\Psi}_a\dot{\Psi}_a \quad \text{and} \quad V = 2\left\{(\Psi_1 - \Psi_2\Psi_3)^2 + (\Psi_2 - \Psi_3\Psi_1)^2 + (\Psi_3 - \Psi_1\Psi_2)^2\right\}. \tag{3.8}$$

The critical points $(\hat{\Psi}_1, \hat{\Psi}_2, \hat{\Psi}_3)$ of this potential are

$$(0,0,0) = \text{minimum}\,, \quad \left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right) = \text{saddle}\,, \quad (\pm1,\pm1,\pm1) = \text{minima}\,, \tag{3.9}$$

with

$$V(\text{minima}) = 0 \qquad \text{and} \qquad V(\text{saddle}) = \frac{3}{8}\,, \tag{3.10}$$

where the number of minus signs in each triple must be even. The central minimum is isotropic with oscillation frequency $\omega = 2$. The other four minima support eigenoscillations with frequencies $\omega_{\|} = 2$ and $\omega_{\perp} = 4$ with respect to the radial direction.

The equations of motion can be obtained either by substituting (3.6) into (3.4) or from (3.7) as the Euler-Lagrange equations,

$$\begin{aligned} \frac{1}{4}\ddot{\Psi}_1 &= -\Psi_1 + 3\,\Psi_2\Psi_3 - \Psi_1(\Psi_2^2 + \Psi_3^2)\,, \\ \frac{1}{4}\ddot{\Psi}_2 &= -\Psi_2 + 3\,\Psi_1\Psi_3 - \Psi_2(\Psi_1^2 + \Psi_3^2)\,, \\ \frac{1}{4}\ddot{\Psi}_3 &= -\Psi_3 + 3\,\Psi_1\Psi_2 - \Psi_3(\Psi_2^2 + \Psi_3^2)\,. \end{aligned} \tag{3.11}$$

These equations are still difficult to solve. However, as can be seen from the contour plot of the potential in figure 1, the system enjoys tetrahedral symmetry. The permutation group S_4 acts on the triple $(\Psi_1, \Psi_2, \Psi_3) \in \mathbb{R}^3$ by permuting the entries and by changing the

[3]This resolves the second equation in (3.4), the first-order Gauß-law constraint. For a more general form of X_a, related with A_k-quivers, see e.g. [15, 16].

[4]Interestingly, $V = \frac{1}{2}\partial_a U \partial_a U$ with a superpotential $U = \Psi_1^2+\Psi_2^2+\Psi_3^2 - 2\Psi_1\Psi_2\Psi_3$, but for Minkowski time this does not yield a gradient flow.

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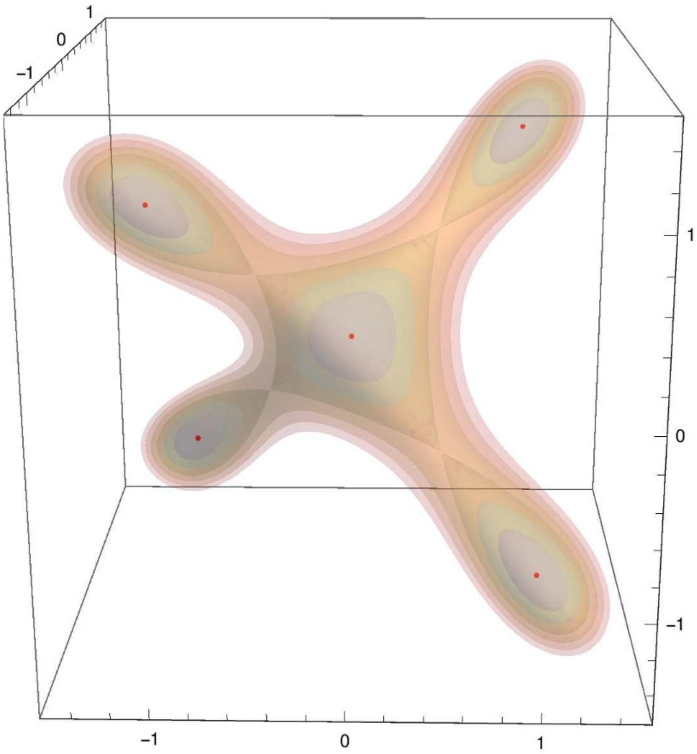

Figure 1. Contours of the Newtonian potential V in (3.8).

sign of an even number of entries. One may hope to find analytic solutions for trajectories fixed under part of this symmetry. The maximal subgroups of S_4 are A_4 (of order 2), D_8 (of order 3) and S_3 (of order 4). While A_4 leaves only the origin invariant, D_8 keeps fixed a coordinate axis (up to sign), and S_3 leaves invariant the direction to a noncentral potential minimum. Therefore, we look at two special cases. In the D_8 case, we pick the Ψ_3-axis and consider

$$\Psi_1 = \Psi_2 = 0 \quad \text{and} \quad \Psi_3 = \xi\,, \tag{3.12}$$

where $\xi(t)$ is some real-valued function of $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$. With this simplification, we get

$$T_\xi = \frac{1}{2}\,\dot{\xi}^2\,, \quad V_\xi = 2\,\xi^2 \qquad \text{and} \qquad \ddot{\xi} = -4\,\xi\,, \tag{3.13}$$

showing that in this direction in parameter space the harmonic approximation is exact. Here, the non-stable S_4 transformations act by permuting the coordinate axes, but two equivalent choices give the same equations. In the S_3 case, we choose the direction $(1,1,1)$ and put

$$\Psi_1 = \Psi_2 = \Psi_3 = \frac{1}{2}(1+\psi)\,, \tag{3.14}$$

where $\psi(t)$ is some other real-valued function of $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$. This ansatz leads to the simplifications

$$T_\psi = \frac{1}{2}\,\dot{\psi}^2\,, \quad V_\psi = \frac{1}{2}\,(1{-}\psi^2)^2 \qquad \text{and} \qquad \ddot{\psi} = 2\,\psi\,(1{-}\psi^2)\,. \tag{3.15}$$

The remaining S_4 transformations flip the sign of two coordinates, which generates three other but equivalent configurations, yielding the same equations. Other directions fixed under some subgroup of S_4 do not give rise to elementary solutions.

Solutions. The general solution of the linear equation (3.13) is

$$\xi(t) = -\frac{\gamma}{2}\cos 2(t-t_0)\,, \tag{3.16}$$

where γ and t_0 are arbitrary real parameters. Since only one $su(2)$ generator is excited, it leads to an Abelian field configuration. We note that the normalization of the linear solution $\xi(t)$ is arbitrary.

For the non-Abelian ansatz (3.14) the simplest solutions of (3.15) are constant at the critical points of V_ψ, i.e.

$$\psi(t) = \pm 1 \ (\text{minima, } V_\psi{=}0) \quad \text{and} \quad \psi(t) = 0 \ \left(\text{local maximum, } V_\psi = \frac{1}{2}\right). \tag{3.17}$$

A prominent nontrivial solution of (3.15) is the bounce,

$$\psi(t) = \sqrt{2}\,\mathrm{sech}\big(\sqrt{2}(t-t_0)\big) = \frac{\sqrt{2}}{\cosh\big(\sqrt{2}(t-t_0)\big)}\,, \tag{3.18}$$

which describes the motion from the local maximum (ψ=0) at $t{=}-\infty$ on the cylinder $\mathbb{R}\times S^3$ to the turning point ($\psi{=}\sqrt{2}$) at $t{=}t_0$ and back to (ψ=0) at $t{=}\infty$. Flipping the sign of $\psi(t)$ produces the anti-bounce, which explores the other half of the double-well potential. In addition, there is a continuum of periodic solutions oscillating either about $\psi = \pm 1$ or exploring both wells of the double-well potential V_ψ, which are given by Jacobi elliptic functions. Usually, the moduli parameter t_0 is trivial because of time translation invariance in (3.15). However, since for de Sitter space according to (2.7) we consider the solutions $\psi(t)$ only on the interval $\mathcal{I} = (-\frac{\pi}{2}, \frac{\pi}{2})$ without imposing boundary conditions, the value of $t_0 \in \mathbb{R}$ makes a difference. It allows us to pick a segment of length π anywhere on the profile of the bounce, not necessarily including its minimum.

Explicit form of the Yang-Mills fields. Let us display the explicit non-Abelian Yang-Mills solutions on dS_4 corresponding to (3.17)–(3.18) after substituting (3.6) and (3.14) into (3.2) and (3.3). For $\psi = \pm 1$ we obtain the trivial solutions $\mathcal{F} \equiv 0$ (vacua). For $\psi = 0$ we get the nontrivial smooth configuration

$$\mathcal{A} = \frac{1}{2}\,e^a\,I_a = \frac{1}{2R}\cos t\,\tilde{e}^a\,I_a = \frac{1}{2R\cosh\tau}\,\tilde{e}^a\,I_a\,, \tag{3.19a}$$

$$\mathcal{F} = -\frac{1}{4}\,\varepsilon^a_{bc}\,e^b{\wedge}e^c\,I_a = -\frac{1}{4R^2}\cos^2\!t\,\varepsilon^a_{bc}\,\tilde{e}^b{\wedge}\tilde{e}^c\,I_a = -\frac{1}{4R^2\cosh^2\!\tau}\,\varepsilon^a_{bc}\,\tilde{e}^b{\wedge}\tilde{e}^c\,I_a\,, \tag{3.19b}$$

where

$$\tilde{e}^0 := \frac{R}{\cos t}\,\mathrm{d}t = R\,\mathrm{d}\tau \qquad \text{and} \qquad \tilde{e}^a := \frac{R}{\cos t}\,e^a = R\,\cosh\tau\,e^a \tag{3.20}$$

constitutes an orthonormal basis for the left-invariant one-forms on de Sitter space. From (3.19b) we read off the color-electric and color-magnetic components

$$\widetilde{E}_a = \frac{\cos^2\!t}{R^2}\,E_a \quad \text{and} \quad \widetilde{B}_a = \frac{\cos^2\!t}{R^2}\,B_a \tag{3.21}$$

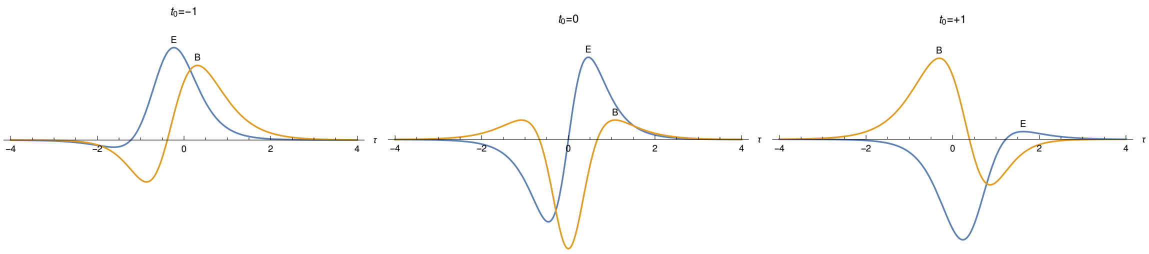

Figure 2. Electric and magnetic amplitudes for the bounce configuration (3.23b) with $t_0{=}{-}1, 0, +1$.

as

$$\tilde{E}_a = \tilde{\mathcal{F}}_{0a} = 0 \quad \text{and} \quad \tilde{B}_a = \frac{1}{2}\varepsilon_{abc}\tilde{\mathcal{F}}_{bc} = -\frac{\cos^2 t}{2R^2}\, I_a = -\frac{1}{2R^2\cosh^2\tau}\, I_a\,. \tag{3.22}$$

with respect to the orthonormal basis (3.20), where we used (2.7). In appendix D we display the explicit coordinate dependence of these field components in different coordinates of S^3.

Inserting the bounce solution (3.18), we obtain a family of nonsingular Yang-Mills configurations,

$$\mathcal{A} = \frac{\cos t}{2R}\left\{1 + \frac{\sqrt{2}}{\cosh\big(\sqrt{2}(t{-}t_0)\big)}\right\}\tilde{e}^a\, I_a\,, \tag{3.23a}$$

$$\mathcal{F} = -\frac{\cos^2 t}{4R^2}\left\{4\frac{\sinh\big(\sqrt{2}(t{-}t_0)\big)}{\cosh^2\big(\sqrt{2}(t{-}t_0)\big)}\,\tilde{e}^0{\wedge}\tilde{e}^a + \frac{\cosh\big(2\sqrt{2}(t{-}t_0)\big)-3}{\cosh^2\big(\sqrt{2}(t{-}t_0)\big)}\,\frac{1}{2}\varepsilon^a_{bc}\,\tilde{e}^b{\wedge}\tilde{e}^c\right\} I_a\,, \tag{3.23b}$$

depending on $t_0 \in \mathbb{R}$. This family carries electric as well as magnetic fields. Their relative size as a function of τ is displayed in figure 2.

Finally, for the Abelian solutions, the substitution of (3.16) yields

$$\mathcal{A} = -\frac{\gamma}{2}\,\cos 2(t{-}t_0)\, e^3\, I_3 = -\frac{\gamma}{2R}\,\cos t\cos 2(t{-}t_0)\,\tilde{e}^3\, I_3\,, \tag{3.24a}$$

$$\mathcal{F} = \mathrm{d}\mathcal{A} = \frac{\gamma}{R^2}\,\cos^2 t\left\{\sin 2(t{-}t_0)\,\tilde{e}^0{\wedge}\tilde{e}^3 + \cos 2(t{-}t_0)\,\tilde{e}^1{\wedge}\tilde{e}^2\right\} I_3\,, \tag{3.24b}$$

hence

$$\widetilde{E}_3 = \widetilde{\mathcal{F}}_{03} = \frac{\gamma}{R^2}\,\cos^2 t\,\sin 2(t{-}t_0)\, I_3\,, \tag{3.25a}$$

$$\widetilde{B}_3 = \widetilde{\mathcal{F}}_{12} = \frac{\gamma}{R^2}\,\cos^2 t\,\cos 2(t{-}t_0)\, I_3\,. \tag{3.25b}$$

Using (2.7), one can rewrite (3.23)–(3.25) in terms of global coordinates (τ,χ,θ,ϕ) on dS_4.

Remark. The Dirac monopole is a connection a_1 (see (B.9)) in the Hopf bundle (B.8) over S^2, with unit topological charge (1st Chern number) given by (B.9). One can embed S^2 in $\mathbb{R}^3$ and lift a_1 from S^2 to $\mathbb{R}^3$. The result is the familiar form of the singular Dirac monopole solution of the Yang-Mills equations on $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$. On the other hand, using the Hopf fibration (B.8) one can pull the monopole connection a_1 back to the 3-sphere S^3 and obtain $e^3 = \pi^* a_1$. Then, the Abelian gauge connection $\mathcal{A}_D = e^3 I_3$ on S^3 is smooth, but it does not satisfy the Yang-Mills equations either on $S^3 \subset \mathrm{dS}_4$ or on dS_4. However, by considering the Abelian potential $\mathcal{A} = \xi(t)\, e^3 I_3$, one obtains the Yang-Mills solution (3.16), (3.24) and (3.25) on dS_4. It oscillates around the Dirac monopole $\mathcal{A}(t{=}t_0) \sim e^3\, I_3$ on S^3.

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Energy of the Yang-Mills solutions. The energy of Yang-Mills configurations on de Sitter space dS_4 computes as

$$\begin{aligned}\mathcal{E} &= -\frac{1}{4}\int_{S^3(t)} \tilde{e}^1\wedge\tilde{e}^2\wedge\tilde{e}^3\ \mathrm{tr}(\widetilde{E}_a\widetilde{E}_a + \widetilde{B}_a\widetilde{B}_a) \\ &= -\frac{\cos t}{4\,R}\int_{S^3} e^1\wedge e^2\wedge e^3\ \mathrm{tr}\left(\mathcal{F}_{0a}\mathcal{F}_{0a} + \frac{1}{2}\,\mathcal{F}_{ab}\mathcal{F}_{ab}\right)\,,\end{aligned} \tag{3.26}$$

where $S^3(t)$ is the 3-sphere of radius $\frac{R}{\cos t}$. On the configuration (3.22), this evaluates to

$$\mathcal{E} = \frac{3\pi^2\,C(j)}{2R\cosh\tau} \tag{3.27}$$

after additional use of (2.7). Similarly, for the configuration (3.23) one gets the *same* result (3.27) as for the purely magnetic configuration. The total energy of the Abelian configuration (3.24) and (3.25) is

$$\mathcal{E}_{\text{Abelian}} = \frac{\pi^2\gamma^2\,C(j)}{2R\cosh\tau}\,, \tag{3.28}$$

where γ^2 is the moduli parameter. We see that, for all these gauge-field configurations, the energy decays exponentially for early and late times. Its finiteness is quite obvious, since our configurations are non-singular on the finite-volume spatial S^3 slices.

Action of the Yang-Mills solutions. In a similar fashion one can evaluate the action functional on the field configurations (3.19b), (3.23b) and (3.24b). Due to conformal invariance, the action functional can be calculated either in the de Sitter metric (2.8) or in the cylinder metric (2.9) on $\mathcal{I}\times S^3$. We have

$$\begin{aligned}S &= \frac{1}{8}\int_{\mathrm{dS}_4} \tilde{e}^0\wedge\tilde{e}^1\wedge\tilde{e}^2\wedge\tilde{e}^3\ \mathrm{tr}(-2\widetilde{\mathcal{F}}_{0a}\widetilde{\mathcal{F}}_{0a} + \widetilde{\mathcal{F}}_{ab}\widetilde{\mathcal{F}}_{ab}) \\ &= \frac{1}{8}\int_{\mathcal{I}\times S^3} e^0\wedge e^1\wedge e^2\wedge e^3\ \mathrm{tr}(-2\mathcal{F}_{0a}\mathcal{F}_{0a} + \mathcal{F}_{ab}\mathcal{F}_{ab})\,,\end{aligned} \tag{3.29}$$

where

$$\mathcal{F} = \frac{1}{2}\,\mathcal{F}_{\mu\nu}\,e^\mu\wedge e^\nu = \frac{1}{2}\,\widetilde{\mathcal{F}}_{\mu\nu}\,\tilde{e}^\mu\wedge\tilde{e}^\nu \qquad \text{for} \quad \mu,\nu = 0,\ldots,3\,, \tag{3.30}$$

and the relation between e^μ and $\tilde{e}^\mu$ is given in (3.20).

For the purely magnetic configuration (3.19b) the action evaluates to

$$S = -\frac{3}{2}\pi^3 C(j)\,. \tag{3.31}$$

One may restore the gauge coupling in the denominator. The action on the 'bounce' configuration (3.23) comes out as

$$\begin{aligned}\frac{S}{C(j)} &= -\frac{3}{2}\pi^3 + 12\,\pi^2\int_{-\pi/2}^{\pi/2}\mathrm{d}t\ \frac{\sinh^2\big(\sqrt{2}(t-t_0)\big)}{\cosh^4\big(\sqrt{2}(t-t_0)\big)} \\ &= -\frac{3}{2}\pi^3 + \sqrt{8}\pi^2\left(\tanh^3\left(\frac{\pi}{\sqrt{2}}+\delta\right) + \tanh^3\left(\frac{\pi}{\sqrt{2}}-\delta\right)\right)\,,\end{aligned} \tag{3.32}$$

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where $\delta = \sqrt{2}\,t_0 \in \mathbb{R}$. Its numerical value varies between 5.52 (for δ=0) and -46.51 (for $\delta \to \pm\infty$). Finally, the action functional on the Abelian solutions (3.24) vanish,

$$
\begin{aligned}
S_{\text{Abelian}} &= -\frac{1}{4}\int_{\mathrm{dS}_4} \tilde{e}^0\wedge\tilde{e}^1\wedge\tilde{e}^2\wedge\tilde{e}^3\ \operatorname{tr}(\widetilde{E}_a\widetilde{E}_a - \widetilde{B}_a\widetilde{B}_a) \\
&= \int_{\mathrm{dS}_4} \tilde{e}^0\wedge\tilde{e}^1\wedge\tilde{e}^2\wedge\tilde{e}^3\ (\tilde{\rho}_e - \tilde{\rho}_m) = 0\,,
\end{aligned}
\tag{3.33}
$$

since the integrals of the electric and magnetic energy densities $\tilde{\rho}_e$ and $\tilde{\rho}_m$ are finite and equal.

In summary, we have described a class of Abelian and non-Abelian gauge configurations solving the Yang-Mills equations on de Sitter space dS_4. They are spatially homogeneous and decay for early and late times. Their energies and actions are all finite.

4 Instantons on de Sitter space $\mathbf{dS_4}$

A useful tool to obtain information about the non-perturbative dynamics of gauge theories in flat space is instanton configurations. It is known that by Euclidean continuation the space dS_4 becomes a 4-sphere S^4 of radius R with the metric (4.2). Therefore, instantons in dS_4 are the standard S^4 instantons. Here we present them in a form adapted to the coordinates on S^4 and the gauge $\mathcal{A}_\varphi = 0$.

Four-sphere. The Euclidean form of the dS_4 metric in global coordinates can be obtained by substituting

$$
\tau = \mathrm{i}\left(\varphi - \frac{\pi}{2}\right) \qquad \text{with} \quad \varphi \in [0,\pi]\,. \tag{4.1}
$$

Then the metric (2.3) becomes the metric on S^4 of radius R,

$$
\mathrm{d}s^2 = R^2(\mathrm{d}\varphi^2 + \sin^2\!\varphi\,\mathrm{d}\Omega_3^2) \qquad \text{with} \quad \mathrm{d}\Omega_3^2 = \delta_{ab}\,e^a e^b\,. \tag{4.2}
$$

This is the standard form in terms of four angles.

By the coordinate transformation

$$
r = R\,\tan\frac{\varphi}{2} \qquad \Longrightarrow \qquad \sin\varphi = \frac{2Rr}{r^2{+}R^2}\,, \qquad \cos\varphi = \frac{R^2{-}r^2}{r^2{+}R^2} \tag{4.3}
$$

it is related to the stereographic coordinates

$$
x^i = r\,\omega^i \qquad \text{for} \quad i=1,\dots,4 \qquad \text{with} \quad r^2 := \delta_{ij}\,x^i x^j\,, \tag{4.4}
$$

so that

$$
\mathrm{d}s^2 = \frac{4R^4}{(r^2{+}R^2)^2}\,\delta_{ij}\,\mathrm{d}x^i\mathrm{d}x^j = \frac{4R^4}{(r^2{+}R^2)^2}\,(\mathrm{d}r^2 + r^2\delta_{ab}\,e^a e^b)\,. \tag{4.5}
$$

Conformal equivalence of metrics on S^4 and $\mathbb{R}\times S^3$. The metric (4.2) or (4.5) is conformally equivalent to the metric on the Euclidean cylinder,

$$\mathrm{d}s^2 = \frac{R^2}{\cosh^2 T}(\mathrm{d}T^2 + \mathrm{d}\Omega_3^2) \tag{4.6}$$

via

$$T = \frac{1}{2}\log\frac{1-\cos\varphi}{1+\cos\varphi} \qquad \Longleftrightarrow \qquad \mathrm{e}^T = \tan\frac{\varphi}{2} \qquad \Longleftrightarrow \qquad \sin\varphi = \frac{1}{\cosh T} \tag{4.7}$$

or

$$T = \log\frac{r}{R} \qquad \Longleftrightarrow \qquad \mathrm{e}^T = \frac{r}{R}\,, \tag{4.8}$$

respectively.

Self-duality. The instanton equations on S^4,

$$\mathcal{F}_{ij} = \frac{1}{2}\sqrt{\det g}\;\varepsilon_{ijkl}\;\mathcal{F}^{kl}\,, \tag{4.9}$$

are conformally invariant, and it is more convenient to consider them on the cylinder $\mathbb{R}\times S^3$ with the metric

$$\mathrm{d}s^2_{\text{cyl}} = \mathrm{d}T^2 + \mathrm{d}\Omega_3^2 = \frac{\cosh^2 T}{R^2}\,\mathrm{d}s^2\,. \tag{4.10}$$

In the basis $(e^i) = (e^a, \mathrm{d}T)$ the SU(2)-invariant (spherically symmetric) connection $\mathcal{A}$ in the gauge $\mathcal{A}_T = 0 = \mathcal{A}_\varphi$ and its curvature are given by [15]

$$\mathcal{A} = X_a\, e^a\,, \qquad \mathcal{F}_{4a} = \frac{\mathrm{d}X_a}{\mathrm{d}T} \qquad \text{and} \qquad \mathcal{F}_{ab} = -2\,\varepsilon_{abc}\,X_c + [X_a, X_b]\,, \tag{4.11}$$

and (4.9) reduces to a form of the generalized Nahm equations given by

$$\frac{\mathrm{d}X_a}{\mathrm{d}T} = 2X_a - \frac{1}{2}\,\varepsilon_{abc}\,[X_b, X_c]\,. \tag{4.12}$$

With the same ansatz as previously,

$$X_1 = \Psi_1\, I_1\,, \quad X_2 = \Psi_2\, I_2\,, \quad X_3 = \Psi_3\, I_3 \qquad \text{with} \quad \Psi_a = \Psi_a(T) \in \mathbb{R} \tag{4.13}$$

these turn into a coupled set of three ordinary first-order differential equations (the dot denotes the derivative with respect to T),

$$\begin{aligned} \frac{1}{2}\dot{\Psi}_1 &= \Psi_1 - \Psi_2\Psi_3 = \frac{1}{2}\frac{\partial U}{\partial \Psi_1}\,, \\ \frac{1}{2}\dot{\Psi}_2 &= \Psi_2 - \Psi_3\Psi_1 = \frac{1}{2}\frac{\partial U}{\partial \Psi_2}\,, \\ \frac{1}{2}\dot{\Psi}_3 &= \Psi_3 - \Psi_1\Psi_2 = \frac{1}{2}\frac{\partial U}{\partial \Psi_3} \end{aligned} \tag{4.14}$$

with the superpotential

$$U = \Psi_1^2 + \Psi_2^2 + \Psi_3^2 - 2\Psi_1\Psi_2\Psi_3\,, \tag{4.15}$$

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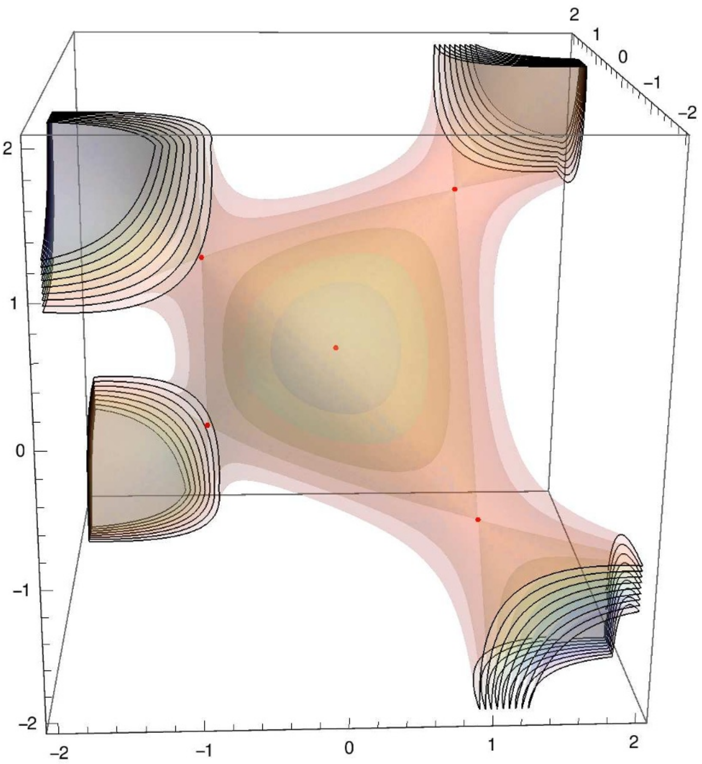

Figure 3. Contours of the superpotential potential U in (4.15).

which is depicted in figure 3. The corresponding Newtonian dynamics is given by

$$\ddot{\Psi}_a = +\frac{\partial V}{\partial \Psi_a} \qquad \text{with} \quad V = \frac{1}{2}\frac{\partial U}{\partial \Psi_a}\frac{\partial U}{\partial \Psi_a} \tag{4.16}$$

yielding the potential given in (3.8), but entering with the opposite sign. Its critical points coincide with the potential minima listed in (3.9), with values $\mathrm{U}(0,0,0) = 0$ and $\mathrm{U}(1,1,1) = 1$. The flow equations (4.14) also imply that

$$\frac{1}{2}\dot{\Psi}_a\dot{\Psi}_a = V(\Psi) \qquad \text{and} \qquad \dot{U} = 2\,V\,. \tag{4.17}$$

Instantons. The static trajectories $\Psi_a = 0$ and $\Psi_a = 1$ (and its images under S_4) lead to the trivial vacuum solution $\mathcal{F} = 0$. However, there exists an analytic BPS solution interpolating between the two kinds of critical points of U. It is captured again by the further simplification

$$\Psi_1 = \Psi_2 = \Psi_3 = \frac{1}{2}(1+\psi) \qquad \text{for} \quad \psi = \psi(T) \in \mathbb{R} \tag{4.18}$$

which leaves us with a single differential equation,

$$\dot{\psi} = 1 - \psi^2 = \frac{\partial U_\psi}{\partial \psi} \qquad \text{with} \quad U_\psi = \psi - \frac{1}{3}\psi^3\,. \tag{4.19}$$

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Its simplest solution is the kink

$$\psi(T) = \tanh(T - T_0) \tag{4.20}$$

with integration constant (or collective coordinate) T_0, which produces

$$X_a = \left[1 + \exp(-2(T - T_0))\right]^{-1} I_a \qquad \text{with} \quad [I_a, I_b] = 2\,\varepsilon^c_{ab} I_c\,. \tag{4.21}$$

By using (4.8), one can rewrite it as

$$X_a = \frac{r^2}{r^2 + \Lambda^2}\, I_a \qquad \text{for} \quad \Lambda^2 := \mathrm{e}^{2T_0} R^2\,, \tag{4.22}$$

which is exactly the BPST instanton extended from $\mathbb{R}^4$ to S^4 [15]. This is easily seen from

$$\mathcal{A} = X_a e^a = -\frac{1}{r^2 + \Lambda^2}\, \eta^a_{ij}\, I_a\, x^i \mathrm{d}x^j \qquad \text{with} \quad e^a = -\frac{1}{r^2}\, \eta^a_{ij}\, x^i\, \mathrm{d}x^j\,, \tag{4.23a}$$

$$\mathcal{F} = -\frac{\Lambda^2}{(r^2 + \Lambda^2)^2}\, \eta^a_{ij} I_a\, \mathrm{d}x^i \wedge \mathrm{d}x^j = -\frac{\Lambda^2 (r^2 + R^2)^2}{4R^4 (r^2 + \Lambda^2)^2}\, \eta^a_{ij} I_a\, \tilde{e}^i \wedge \tilde{e}^j \quad \text{for} \quad \tilde{e}^i = \frac{2R^2\, \mathrm{d}x^i}{r^2 + R^2}\,, \tag{4.23b}$$

where the $\tilde{e}^i$ form an orthonormal basis of one-forms on S^4. For $R = \Lambda$ ($T_0 = 0$), $\mathcal{F}$ has the canonical form of BPST instanton on S^4. The radius R of S^4 sets the scale for Λ, but we may tune T_0 in (4.22) such as to remove the dependence of Λ on R. The action of this configuration evaluates to $S = 8\pi^2$ independent of R. The anti-instanton is found by flipping the sign of T.

Remark. Of course, the Newton equation (4.16) has more solutions than the flow equations (4.14). For example, other bounded solutions for ψ oscillate anharmonically between $\psi = -1$ and $\psi = 1$. When viewed in the full parameter space $\mathbb{R}^3 \ni (\Psi_a)$ however, almost all classical trajectories will run away to infinity, since the inverted potential $-V$ does not have any local minimum. This is reflected in the value of the topological charge

$$\begin{aligned} q &= -\frac{1}{64\pi^2 C(j)} \int_{\mathbb{R} \times S^3} \mathrm{d}T \wedge e^1 \wedge e^2 \wedge e^3\, \varepsilon^{ijkl}\, \mathrm{tr}(\mathcal{F}_{ij} \mathcal{F}_{kl}) \\ &= \int \mathrm{d}T\, \left(2(\Psi_1 - \Psi_2 \Psi_3)\dot{\Psi}_1 \,+\, \text{cyclic}\right) = \int \mathrm{d}T\, \dot{U} \end{aligned} \tag{4.24}$$

which differs from zero or infinity only if the trajectory $\Psi_a(T)$ connects the two types of critical points, i.e. for an instanton or anti-instanton. Those two saturate the inequality $S \geq 4\pi^2 C(j)\, |q|$.

5 Description of anti-de Sitter space AdS_4

AdS_3-slicing coordinates. For the remainder of the paper we attempt to repeat the previous analysis for *anti*-de Sitter space AdS_4. In analogy with the dS$_4$ case, where we used the fact that $S^3 \cong \mathrm{SU}(2)$ is a group manifold, for AdS_4 we may employ instead another group manifold,

$$\mathrm{AdS}_3 \;\cong\; \mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm \mathbf{1}_2\}\,, \tag{5.1}$$

and embed AdS_4 into $\mathbb{R}^{3,2}$ in such a way that the metric on AdS_4 will be conformally equivalent to the metric on a cylinder $\mathbb{R}\times \mathrm{PSL}(2,\mathbb{R})$, in order to follow our recipe for constructing Yang-Mills solutions.

So, $\mathrm{AdS}_4 \cong \mathrm{O}(3,2)/\mathrm{O}(3,1)$ is a hypersurface in $\mathbb{R}^{3,2}$ topologically equivalent to $S^1\times\mathbb{R}^3$ and defined by

$$(y^1)^2+(y^2)^2-(y^3)^2-(y^4)^2+(y^5)^2=-R^2\,. \tag{5.2}$$

One can introduce global coordinates (z,t,ρ,ϕ) by setting

$$y^i=R\,\omega^i\cosh z\,,\quad y^5=R\,\sinh z\qquad\text{with}\quad z\in\mathbb{R}\quad\text{and}\quad \eta_{ij}\,\omega^i\omega^j=-1 \tag{5.3}$$

for $\omega^i = \omega^i(t,\rho,\phi)$ with $i=1,\dots,4$ embedding AdS_3 into $\mathbb{R}^{2,2}$ with metric $(\eta_{ij})=\mathrm{diag}(1,1,-1,-1)$. A dimensional coordinate $\tilde{z}$ can be introduced as $\tilde{z}=R\,z$. The flat metric on $\mathbb{R}^{3,2}$ induces a metric on AdS_4,

$$\mathrm{d}s^2=R^2\left(\mathrm{d}z^2+\cosh^2\!z\,\mathrm{d}\Omega^2_{2,1}\right), \tag{5.4}$$

where $\mathrm{d}\Omega^2_{2,1}$ denotes the metric on the unit-radius $\mathrm{AdS}_3\cong\mathrm{PSL}(2,\mathbb{R})$.

On this space we introduce an orthonormal basis $\{e^\alpha\}$, $\alpha=0,1,2$, of left-invariant one-forms which satisfy the equations

$$\mathrm{d}e^\alpha+f^\alpha_{\beta\gamma}\,e^\beta\wedge e^\gamma=0\,, \tag{5.5}$$

where

$$f^\alpha_{\beta\gamma}=\eta^{\alpha\delta}\varepsilon_{\delta\beta\gamma}\qquad\text{for}\quad(\eta_{\alpha\beta})=\mathrm{diag}(-1,+1,+1) \tag{5.6}$$

are the structure constants of the group $\mathrm{SL}(2,\mathbb{R})$. Concretely, from (5.6) we have

$$f^1_{20}=f^2_{01}=1\quad\text{and}\quad f^0_{12}=-1\qquad\text{for}\quad\varepsilon_{012}=1\,. \tag{5.7}$$

In terms of e^α the AdS_3 metric has the form

$$\mathrm{d}\Omega^2_{2,1}=\eta_{\alpha\beta}\,e^\alpha e^\beta=-(e^0)^2+(e^1)^2+(e^2)^2\,. \tag{5.8}$$

Explicit formulæ for coordinates and one-forms on unit AdS_3 can be found in appendix C.

Conformal coordinates I. Instead of the coordinate z one can introduce the coordinate

$$\chi=\arctan(\sinh z)\quad\Longleftrightarrow\quad\sinh z=\tan\chi\,,\quad\cosh z=\frac{1}{\cos\chi} \tag{5.9}$$

in which $z\in(-\infty,\infty)$ corresponds to $\chi\in(-\frac{\pi}{2},\frac{\pi}{2})$. The metric (5.4) in the coordinates (χ,t,ρ,ϕ) reads

$$\mathrm{d}s^2=\frac{R^2}{\cos^2\!\chi}\left(\mathrm{d}\chi^2+\mathrm{d}\Omega^2_{2,1}\right)=\frac{R^2}{\cos^2\!\chi}\,\mathrm{d}s^2_{\mathrm{cyl}}\,, \tag{5.10}$$

where

$$\mathrm{d}s^2_{\mathrm{cyl}}=\mathrm{d}\chi^2+\eta_{\alpha\beta}\,e^\alpha e^\beta=\eta_{\mu\nu}\,e^\mu e^\nu\qquad\text{for}\quad\mu,\nu,=0,\dots,3\quad\text{and}\quad e^3:=\mathrm{d}\chi \tag{5.11}$$

is the metric on the cylinder $\mathbb{R} \times \mathrm{AdS}_3$ with the Minkowski metric

$$(\eta_{\mu\nu}) = \mathrm{diag}(\eta_{\alpha\beta}, +1) = \mathrm{diag}(-1, +1, +1, +1) \tag{5.12}$$

in the orthonormal basis (e^μ). Hence, we see that anti-de Sitter space is conformally equivalent to the finite cylinder $\mathcal{I}\times \mathrm{PSL}(2,\mathbb{R})$ with the interval $\mathcal{I} = (-\frac{\pi}{2}, \frac{\pi}{2})$, fully parallel to de Sitter space after substituting $\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{AdS}_3$ for of $\mathrm{SU}(2)\cong S^3$ and switching the signature of the cylinder coordinate.

H^3-slicing coordinates. Anti-de Sitter space is the one-sheeted hyperboloid embedded in flat $\mathbb{R}^{3,2}$ by the relation (5.2). Another natural slicing is provided by the global coordinates (t, ρ, θ, ϕ)

$$y^1 = R\sinh\rho\,\lambda^2\,,\ \ y^2 = R\sinh\rho\,\lambda^1\,,\ \ y^3 = R\cosh\rho\,\cos t\,,\ \ y^4 = R\cosh\rho\,\sin t\,,\ \ y^5 = R\sinh\rho\,\lambda^3\,, \tag{5.13}$$

where $t \in [-\pi, \pi)$ parametrizes a circle, $\rho \geq 0$, and $\lambda^a = \lambda^a(\theta, \phi)$ from (2.12) embed S^2 into $\mathbb{R}^3$ in the standard manner. The flat metric on $\mathbb{R}^{3,2}$ induces on AdS_4 the metric

$$\mathrm{d}s^2 = R^2\big(-\cosh^2\!\rho\,\mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2\!\rho\,\mathrm{d}\Omega_2^2\big) \qquad \text{with} \quad \mathrm{d}\Omega_2^2 = \mathrm{d}\theta^2 + \sin^2\!\theta\mathrm{d}\phi^2\,, \tag{5.14}$$

showing that equal-time slices are hyperbolic 3-spaces H^3. A dimensionful time coordinate $\tilde{t}$ can be introduced as $\tilde{t} = Rt$ with $\tilde{t} \in [-\pi R, \pi R)$. The conformal boundary $\rho \to \infty$ for this metric has topology $S^1 \times S^2$ with coordinates (t, θ, ϕ). One can unwrap the circle $S^1 = \mathbb{R}/\mathbb{Z}$ and extend the time coordinate t to all of $\mathbb{R}$, which means considering the universal covering space $\widetilde{\mathrm{AdS}}_4$ of AdS_4 having topology $\mathbb{R}^4$ instead of $S^1 \times \mathbb{R}^3$.

Conformal coordinates II. Instead of the coordinate ρ in (5.13) and (5.14) one can introduce the coordinate [17][5]

$$\chi = \arctan(\sinh\rho) \quad\iff\quad \sinh\rho = \tan\chi\,, \quad \cosh\rho = \frac{1}{\cos\chi}\,, \tag{5.15}$$

in which $\rho \in [0, \infty)$ corresponds to $\chi \in [0, \frac{\pi}{2})$. The metric (5.14) in the coordinates (t, χ, θ, ϕ) reads

$$\mathrm{d}s^2 = \frac{R^2}{\cos^2\!\chi}\big(-\mathrm{d}t^2 + \mathrm{d}\chi^2 + \sin^2\!\chi\,\mathrm{d}\Omega_2^2\big) = \frac{R^2}{\cos^2\!\chi}\big(-\mathrm{d}t^2 + \mathrm{d}\Omega_{3+}^2\big) \tag{5.16}$$

where $\mathrm{d}\Omega_{3+}^2$ is the metric on the upper hemisphere S^3_+ of the 3-sphere $S^3 = S^3_+ \cup S^2 \cup S^3_-$, since the conformal boundary $\rho \to \infty$ has been retracted to the finite boundary at $\chi{=}\frac{\pi}{2}$ corresponding to the equator of S^3 for any value of t. The metric $\mathrm{d}\Omega_{3+}^2$ differs from the S^3 metric in (B.2) only by the range of χ ($\frac{\pi}{2}$ rather than π). Both patches $S^3_\pm$, introduced in (2.10), have the topology of $\mathbb{R}^3$. However, the metric on S^3_+ is the standard metric on the 3-sphere and can be written as

$$\mathrm{d}\Omega_{3+}^2 = \delta_{ab}\, e^a e^b\,, \tag{5.17}$$

where the e^a are defined in (2.4)–(2.6) but are considered only on the upper hemisphere.

[5] This coordinate χ differs from χ in (5.9) but agrees with (half of) χ parametrizing S^3 in (B.2).

From (5.16) we see that the metric on anti-de Sitter space is also conformally equivalent to

$$\mathrm{d}s^2_{\mathrm{cyl}} = -\mathrm{d}t^2 + \delta_{ab}\, e^a e^b = \eta_{\mu\nu}\, e^\mu e^\nu \qquad \text{for} \quad \mu,\nu = 0,\dots,3 \quad \text{and} \quad e^0 := \mathrm{d}t\,, \tag{5.18}$$

where

$$t \in [-\pi, \pi) \quad \text{for} \quad \mathrm{AdS}_4 \qquad \text{and} \qquad t \in \mathbb{R} \quad \text{for} \quad \widetilde{\mathrm{AdS}}_4\,, \tag{5.19}$$

so we are dealing with $S^1 \times S^3_+ \cong S^1 \times \mathbb{R}^3$ or with a Lorentzian cylinder $\mathbb{R} \times S^3_+ \cong \mathbb{R}^4$, respectively. One should be clear about which space, AdS_4 or $\widetilde{\mathrm{AdS}}_4$, is considered.

6 Yang-Mills configurations on AdS_4

Here we describe some Yang-Mills configurations on AdS_4 with the metric (5.10) conformally equivalent to the cylinder metric (5.11) on $\mathcal{I} \times \mathrm{AdS}_3$ and group structure on $\mathrm{AdS}_3 \cong \mathrm{PSL}(2,\mathbb{R})$ discussed in section 5. Of course, the list of solutions we construct is not exhaustive. First we will consider solutions which are naturally described in the metrics (5.10) and (5.11). Their energy and action are infinite due to infinite volume of the space $\mathrm{AdS}_3 \cong \mathrm{PSL}(2,\mathbb{R})$. Then we will find solutions naturally described in the metric (5.16)–(5.18) and will show that both their energy and action are *finite* on AdS_4.

Similar to section 3, solutions on AdS_4 can be obtained by solving the Yang-Mills equations on $\mathcal{I} \times \mathrm{PSL}(2,\mathbb{R})$ with the metric (5.11) or on $S^1 \times S^3_+$ with the metric (5.18). And again, on $\mathrm{SU}(N)$-valued gauge-group elements $g(\cdot)$, acting on $su(N)$-valued gauge fields $\mathcal{A}$ and $\mathcal{F}$, we can impose the boundary condition $g(\partial(\mathcal{I} \times \mathrm{AdS}_3)) = \mathrm{Id}$ on the boundary $\partial(\mathcal{I} \times \mathrm{AdS}_3) = \mathrm{AdS}_3|_{\chi=\pm\pi/2} = \mathrm{AdS}_3|_{z=\pm\infty}$ and similarly $g = \mathrm{Id}$ on the boundary $\partial(\mathcal{T} \times S^3_+) = \mathcal{T} \times S^2$ for the metric (5.18), where $\mathcal{T} = S^1$ for AdS_4 and $\mathcal{T} = \mathbb{R}$ for $\widetilde{\mathrm{AdS}}_4$.

Matrix equations. We first employ the metrics (5.10) and (5.11). We will be concise in our discussion since all our steps will repeat those from section 3. For $su(N)$-valued gauge potentials $\mathcal{A}$ in the gauge $\mathcal{A}_3 = \mathcal{A}_\chi = 0 = \mathcal{A}_z$ we employ the ansatz

$$\mathcal{A} = X_\alpha(\chi)\, e^\alpha\,. \tag{6.1}$$

Here, $X_\alpha(\chi)$ are three $su(N)$-valued functions depending only on $\chi \in \mathcal{I}$, and e^α are one-forms on $\mathrm{PSL}(2,\mathbb{R})$ given in (C.3) and satisfying (5.5). The field strength for this ansatz reads

$$\mathcal{F} = \mathcal{F}_{3\alpha}\, e^3 \wedge e^\alpha + \frac{1}{2}\mathcal{F}_{\alpha\beta}\, e^\alpha \wedge e^\beta = X'_\alpha\, e^3 \wedge e^\alpha + \frac{1}{2}(-2f^\alpha_{\beta\gamma} X_\alpha + [X_\beta, X_\gamma])\, e^\beta \wedge e^\gamma\,, \tag{6.2}$$

where $X'_\alpha := \mathrm{d}X_\alpha/\mathrm{d}\chi$ and $e^\mu = (e^\alpha, e^3) = (e^\alpha, \mathrm{d}\chi)$ for $\alpha = 0,1,2$. After substitution of (6.1) and (6.2), the Yang-Mills equations on $\mathcal{I} \times \mathrm{PSL}(2,\mathbb{R})$ reduce to the matrix differential equations

$$\begin{aligned} X''_0 &= -4X_0 - 6[X_1, X_2] + \big[X_1, [X_0, X_1]\big] + \big[X_2, [X_0, X_2]\big]\,, \\ X''_1 &= -4X_1 + 6[X_2, X_0] + \big[X_2, [X_1, X_2]\big] - \big[X_0, [X_1, X_0]\big]\,, \\ X''_2 &= -4X_2 + 6[X_0, X_1] - \big[X_0, [X_2, X_0]\big] + \big[X_1, [X_2, X_1]\big]\,, \end{aligned} \tag{6.3}$$

where $X''_\alpha := \mathrm{d}^2 X_\alpha/\mathrm{d}\chi^2$. Comparison to (3.4) shows that the two sets of equations are related by[6]

$$X_3 \;\mapsto\; X_0\,, \qquad X_1 \;\mapsto\; \pm\mathrm{i}\,X_1\,, \qquad X_2 \;\mapsto\; \pm\mathrm{i}\,X_2\,, \tag{6.4}$$

reflecting the relation between the SU(2) and SL(2,$\mathbb{R}$) generators.

Reduction to particle mechanics. As in section 3, we take the matrices X_α from a spin-j representation of SU(2) with generators (I_1, I_2, I_3) inside $su(N)$ with $N = 2j + 1$ and put

$$X_0 = \Psi_0\, I_3\,, \quad X_1 = \Psi_1\, I_1 \quad \text{and} \quad X_2 = \Psi_2\, I_2\,, \tag{6.5}$$

where Ψ_α are real functions of χ. Substituting (6.5) into (6.3), we obtain

$$\begin{aligned}
\tfrac{1}{4}\Psi''_0 &= -\Psi_0 - 3\,\Psi_1\Psi_2 + \Psi_0(\Psi_1^2+\Psi_2^2) = -\frac{\partial V}{\partial \Psi_0}\,,\\
\tfrac{1}{4}\Psi''_1 &= -\Psi_1 + 3\,\Psi_2\Psi_0 - \Psi_1(\Psi_0^2-\Psi_2^2) = +\frac{\partial V}{\partial \Psi_1}\,,\\
\tfrac{1}{4}\Psi''_2 &= -\Psi_2 + 3\,\Psi_0\Psi_1 - \Psi_2(\Psi_0^2-\Psi_1^2) = +\frac{\partial V}{\partial \Psi_2}
\end{aligned} \tag{6.6}$$

for a quasi-potential function

$$V = 2\left\{(\Psi_0+\Psi_1\Psi_2)^2 - (\Psi_1-\Psi_2\Psi_0)^2 - (\Psi_2-\Psi_0\Psi_1)^2\right\}, \tag{6.7}$$

consistent with (6.4). However, the interpretation of a Newtonian dynamics is disturbed by the fact that the quasi-kinetic energy

$$T = \frac{1}{2}\,(\dot\Psi_0^2 - \dot\Psi_1^2 - \dot\Psi_2^2) = -\frac{1}{2}\,\eta^{\alpha\beta}\,\dot\Psi_\alpha\dot\Psi_\beta \tag{6.8}$$

inherits the indefiniteness of the AdS$_3$ metric, giving a negative 'mass' to Ψ_1 and Ψ_2.

The Yang-Mills Lagrangian on the cylinder $\mathcal{I}\times\mathrm{PSL}(2,\mathbb{R})$ becomes

$$\mathcal{L} = \frac{1}{8}\,\mathrm{tr}\,\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} = 2\,C(j)\,(T-V)\,, \tag{6.9}$$

and the Euler-Lagrange equations derived from (6.9) coincide with (6.6).

Solutions. The system (6.6) is invariant only under a D_8 subgroup of the tetrahedral symmetry of (3.11). Therefore, the Abelian dS$_4$ solution (3.12) also applies here,

$$\Psi_1 = \Psi_2 = 0 \quad \text{and} \quad \Psi_0 = \xi \quad \Rightarrow \quad \xi'' = -4\,\xi \quad \Rightarrow \quad \xi(\chi) = \frac{\gamma}{2}\,\cos 2(\chi-\chi_0)\,, \tag{6.10}$$

where γ and χ_0 are arbitrary real parameters. Analogous solutions exist exciting only Ψ_1 or Ψ_2.

We cannot write down nontrivial analytic solutions of the system (6.6). In particular, there is no analog of the bounce solution (3.18) to (3.14) and (3.15). However, static solutions exist, since the quasi-potential (6.7) has the 5 critical points $(\hat\Psi_0, \hat\Psi_1, \hat\Psi_2)$ with values

$$(0,0,0) \;\Rightarrow\; V=0\,, \qquad (\pm3,\pm1,\pm1) \;\Rightarrow\; V=16\,, \tag{6.11}$$

[6]The sign choice for X_1 and for X_2 must be the same.

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where the number of minus signs in each triple must be even. The $\Psi_\alpha{=}0$ configuration corresponds to the vacuum solution $\mathcal{F}{=}0$, while the other critical points yield genuine non-Abelian Yang-Mills solutions.

Yang-Mills solutions with infinite action. Let us display the explicit form of Yang-Mills configurations on AdS_4 corresponding to the solutions (6.10) and (6.11) of the reduced Yang-Mills equations (6.6). Computing their energy and action entails integrating over the spatial part of the AdS_3 slice, which is the hyperbolic space H^2. Since the latter has infinite volume, as can be seen from the metric (C.2), the energy and the action of these solutions are infinite.

Substituting $(\Psi_0,\Psi_1,\Psi_2)=(3,1,1)$ into (6.5), (6.1) and (6.2), we obtain the solution

$$\mathcal{A}=\frac{1}{R}\cos\chi\left(3\,I_3\,\tilde{e}^0+I_1\,\tilde{e}^1+I_2\,\tilde{e}^2\right)\quad\text{and}\quad\mathcal{F}=\frac{4}{R^2}\cos^2\!\chi\left(2\,I_3\,\tilde{e}^1\wedge\tilde{e}^2+I_2\,\tilde{e}^0\wedge\tilde{e}^1+I_1\,\tilde{e}^2\wedge\tilde{e}^0\right), \tag{6.12}$$

where $\tilde{e}^\mu=R\,e^\mu/\cos\chi$ for $(\mu)=(\alpha,3)$ is the orthonormal basis on AdS_4 for the metric (5.10). We read off the color-electric and color-magnetic components

$$\begin{aligned}\widetilde{E}_1&=\widetilde{\mathcal{F}}_{01}=\frac{4}{R^2\cosh^2\!z}\,I_2\,,\\ \widetilde{E}_2&=\widetilde{\mathcal{F}}_{02}=\frac{-4}{R^2\cosh^2\!z}\,I_1\,,\\ \widetilde{B}_3&=\widetilde{\mathcal{F}}_{12}=\frac{8}{R^2\cosh^2\!z}\,I_3\,,\end{aligned} \tag{6.13}$$

where we used the relations (5.9). All other components vanish since $\widetilde{\mathcal{F}}_{\alpha3}=0$. Flipping two of the signs in the solution (Ψ_0,Ψ_1,Ψ_2) produces analogous configurations, which differ from (6.12) and (6.13) only by switching the signs of two in three terms correspondingly.

Substituting (6.10) into (6.5), (6.1) and (6.2), we get the Abelian solution

$$\mathcal{A}=\frac{\gamma}{2}\,\cos 2(\chi{-}\chi_0)\,e^0\,I_3=\frac{\gamma}{2R}\,\cos\chi\,\cos 2(\chi{-}\chi_0)\,\tilde{e}^0\,I_3\,, \tag{6.14a}$$

$$\mathcal{F}=\mathrm{d}\mathcal{A}=\frac{\gamma}{R^2}\,\cos^2\!\chi\left\{\sin 2(\chi{-}\chi_0)\,\tilde{e}^0\wedge\tilde{e}^3+\cos 2(\chi{-}\chi_0)\,\tilde{e}^1\wedge\tilde{e}^2\right\}I_3 \tag{6.14b}$$

and therefore

$$\widetilde{E}_3=\widetilde{\mathcal{F}}_{03}=\frac{\gamma}{R^2}\,\cos^2\!\chi\,\sin 2(\chi{-}\chi_0)\,I_3\,, \tag{6.15a}$$

$$\widetilde{B}_3=\widetilde{\mathcal{F}}_{12}=\frac{\gamma}{R^2}\,\cos^2\!\chi\,\cos 2(\chi{-}\chi_0)\,I_3\,. \tag{6.15b}$$

Using the correspondence (5.9) one can rewrite (6.14) and (6.15) in terms of the z coordinate used in (5.4). For the Abelian solution, the action is proportional to

$$\mathrm{vol}\,(\mathrm{PSL}(2,\mathbb{R}))\times\int_{-\pi/2}^{\pi/2}\mathrm{d}\chi\,\left(\sin^2 2(\chi{-}\chi_0)-\cos^2 2(\chi{-}\chi_0)\right). \tag{6.16}$$

The above integral vanishes but it multiplies the infinite group volume.[7]

[7] We may regularize the volume of AdS_3 before integrating over χ and thus obtain a vanishing action in this case.

Yang-Mills solutions with finite action. Now we consider the metric (5.16) on AdS$_4$. Thanks to the conformal invariance of the Yang-Mills equations in four dimensions, it suffices to study Yang-Mills theory on $S^1\times S^3_+$ with the metric (5.18) including the standard metric (B.2) restricted to the upper hemisphere S^3_+. Since the one-forms e^a in (5.17) obey (2.4), we can literally copy the ansatz (3.2) for $\mathcal{I}\times S^3$ to our space $S^1\times S^3_+$,

$$\mathcal{A} = X_a(t)\, e^a\,. \tag{6.17}$$

Then all formulæ (3.4)–(3.17) are valid in this case as well, yielding the same matrix equations, three-dimensional Newtonian dynamics and its solutions as in the de Sitter case. However, the periodicity in t in addition requires

$$\Psi_a(t{+}2\pi) = \Psi_a(t)\,. \tag{6.18}$$

Consequently, the constant and periodic Yang-Mills solutions (3.19) and (3.24) on dS$_4$ are also valid on AdS$_4$, after changing their conformal factor,

$$\tilde e^\mu = \frac{R}{\cos t}\, e^\mu \qquad \Rightarrow \qquad \tilde e^\mu = \frac{R}{\cos\chi}\, e^\mu \qquad \text{with} \quad e^0 = \mathrm{d}t\,, \tag{6.19}$$

and restricting e^a to the upper hemisphere S^3_+. The bounce solution (3.18) does not qualify. However, it is the limiting case of a continuum of periodic solutions given by a Jacobi elliptic function,

$$\psi(t;k) = \alpha(k)\,\mathrm{dn}[\alpha(k)(t{-}t_0);k] \qquad \text{with} \quad \alpha(k)=\sqrt{2/(2{-}k^2)} \quad \text{and} \quad 0\le k<1\,. \tag{6.20}$$

This family interpolates between the bounce (3.18) for $k\to 1$ (period $\to\infty$) and the constant vacuum solution $\psi=1$ for $k=0$. For infinitesimally small values of k we have harmonic oscillations with a period of π, as can be gleaned from a harmonic approximation to (3.15). By continuity, shorter periods cannot be attained, but there should exist a 2π-periodic solution for a special value of k. Since $\mathrm{dn}[u;k]$ has a period of $2\mathcal{K}(k)$, where $\mathcal{K}(k)$ is the complete elliptic integral of the first kind (see e.g. the appendix of [14] for a brief discussion of Jacobi functions), the periodicity condition is satisfied if [18, 19]

$$\frac{\mathcal{K}(k)}{\alpha(k)} = \pi \qquad \Rightarrow \qquad k=\bar k \approx 0.9977\,, \tag{6.21}$$

which is very near to the bounce, as can be seen in figure 4.

We lift all these solutions from the cylinder (5.18) to AdS$_4$ with metric (5.16) written as

$$\mathrm{d}s^2 = \frac{R^2}{\cos^2\!\chi}\,(-\mathrm{d}t + \mathrm{d}\Omega^2_{3+}) = \eta_{\mu\nu}\,\tilde e^\mu \tilde e^\nu \tag{6.22}$$

and obtain

$$\mathcal{A} = \frac12\, e^a\, I_a = \frac{1}{2R}\cos\chi\, \tilde e^a\, I_a\,, \tag{6.23a}$$

$$\mathcal{F} = -\frac14\,\varepsilon^a_{bc}\, e^b\wedge e^c\, I_a = -\frac{1}{4R^2}\cos^2\!\chi\,\varepsilon^a_{bc}\,\tilde e^b\wedge\tilde e^c\, I_a \tag{6.23b}$$

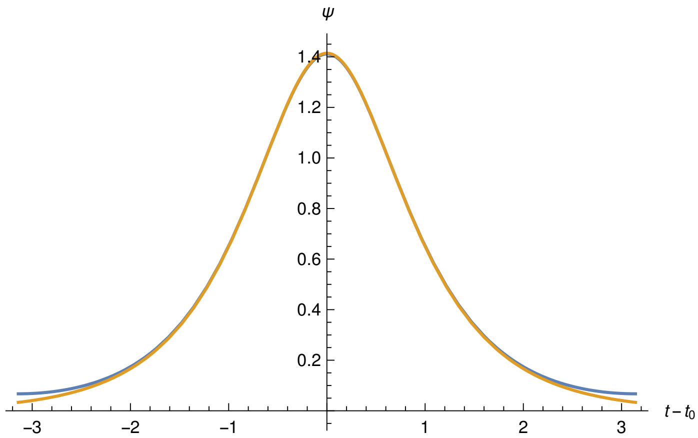

Figure 4. Bounce (yellow) and 2π-periodic (blue) solution in the double-well potential (3.15).

for the colour-magnetic solution,

$$\mathcal{A} = \frac{1}{2\,R}\,\cos\chi\,\big(1+\alpha(\bar{k})\,\mathrm{dn}[\dots]\big)\,\tilde{e}^a\,I_a\,, \tag{6.24a}$$

$$\mathcal{F} = \frac{1}{2\,R^2}\,\cos^2\!\chi\Big\{\bar{k}\,\alpha(\bar{k})^2\,\mathrm{cn}[\dots]\,\mathrm{sn}[\dots]\,\tilde{e}^0\!\wedge\tilde{e}^a - \frac{1}{2}\,\big(1-\alpha(\bar{k})^2\,\mathrm{dn}^2[\dots]\big)\,\varepsilon^a_{bc}\,\tilde{e}^b\!\wedge\tilde{e}^c\Big\}I_a \tag{6.24b}$$

for the near-bounce solution, with arguments $[\dots] = [\alpha(\bar{k})(t-t_0)]$ for Jacobi elliptic functions cn, sn and dn, and

$$\mathcal{A} = -\frac{\gamma}{2}\,\cos 2(t-t_0)\,e^3\,I_3 = -\frac{\gamma}{2\,R}\,\cos\chi\,\cos 2(t-t_0)\,\tilde{e}^3\,I_3\,, \tag{6.25a}$$

$$\mathcal{F} = \mathrm{d}\mathcal{A} = \frac{\gamma}{R^2}\,\cos^2\!\chi\,\big\{\sin 2(t-t_0)\,\tilde{e}^0\!\wedge\tilde{e}^3 + \cos 2(t-t_0)\,\tilde{e}^1\!\wedge\tilde{e}^2\big\}\,I_3 \tag{6.25b}$$

for the Abelian solution which stems from (3.16). Using (5.15), one can rewrite (6.23)–(6.25) in terms of coordinates (t,ρ,θ,ϕ) on AdS_4.

Energy of the Yang-Mills solutions. The energy of Yang-Mills configurations on anti-de Sitter space AdS_4 computes as

$$\begin{aligned}\mathcal{E} &= -\frac{1}{4}\int_{\tilde{S}^3_+}\tilde{e}^1\!\wedge\tilde{e}^2\!\wedge\tilde{e}^3\;\mathrm{tr}\left(\widetilde{\mathcal{F}}_{0a}\widetilde{\mathcal{F}}_{0a} + \frac{1}{2}\,\widetilde{\mathcal{F}}_{ab}\widetilde{\mathcal{F}}_{ab}\right)\\ &= -\frac{1}{4\,R}\int_{S^3_+} e^1\!\wedge e^2\!\wedge e^3\;\cos\chi\;\mathrm{tr}\left(\mathcal{F}_{0a}\mathcal{F}_{0a} + \frac{1}{2}\,\mathcal{F}_{ab}\mathcal{F}_{ab}\right)\,,\end{aligned} \tag{6.26}$$

where $\tilde{S}^3_+$ is the hemisphere with its metric conformally rescaled by $\frac{R^2}{\cos^2\chi}$.

On the configuration (6.23), the energy evaluates to

$$\mathcal{E} = \frac{\pi\,C(j)}{R}\,. \tag{6.27}$$

We see that it is not only finite but, contrary to the de Sitter case, it does not depend on time. Hence, we get a static magnetic configuration. For the configuration (6.24) we use results about sphalerons on a circle [18, 19] to calculate the energy and find

$$\mathcal{E}(k) = \frac{\pi\, C(j)}{R}\, \frac{4\, k^2}{(1+k^2)^2}\,, \tag{6.28}$$

which correctly interpolates between the values for the vacuum ($k = 0$) and the static magnetic solution ($k \to 1$). For the admissible value of k, its value $\mathcal{E}(\bar{k})$ differs from (6.27) by a factor of about $1 - 5{\times}10^{-6}$. Finally, the energy of the Abelian configuration (6.25) is

$$\mathcal{E}_{\text{Abelian}} = \frac{\pi\, \gamma^2\, C(j)}{3\, R}\,, \tag{6.29}$$

where γ^2 is the moduli parameter. So, for all three Yang-Mills solutions the energy is finite and constant.

Action of the Yang-Mills solutions. As previously, the action functional on the field configurations (6.23)–(6.25) can be calculated either in the anti-de Sitter metric (5.16) or in the cylinder metric (5.18) on $S^1{\times}S^3_+$,

$$\begin{aligned} S &= \frac{1}{8}\int_{\text{AdS}_4} \tilde{e}^0{\wedge}\tilde{e}^1{\wedge}\tilde{e}^2{\wedge}\tilde{e}^3\ \operatorname{tr}(-2\widetilde{\mathcal{F}}_{0a}\widetilde{\mathcal{F}}_{0a} + \widetilde{\mathcal{F}}_{ab}\widetilde{\mathcal{F}}_{ab}) \\ &= \frac{1}{8}\int_{S^1\times S^3_+} e^0{\wedge}e^1{\wedge}e^2{\wedge}e^3\ \operatorname{tr}(-2\mathcal{F}_{0a}\mathcal{F}_{0a} + \mathcal{F}_{ab}\mathcal{F}_{ab})\,. \end{aligned} \tag{6.30}$$

For the purely magnetic configuration (6.23) the action evaluates to

$$S = -\frac{3}{2}\,\pi^3 C(j)\,. \tag{6.31}$$

Interestingly, it coincides with the value (3.31) for the analogous but non-static configuration on dS_4. This is because they are identical but are lifted from different spaces which happen to have the same volume,

$$\operatorname{vol}(\mathcal{I}{\times}S^3) = \pi \times 2\pi^2 = 2\pi \times \pi^2 = \operatorname{vol}(S^1{\times}S^3_+)\,. \tag{6.32}$$

The action of the configuration (6.24) is reduced to

$$S(k) = \frac{3}{4}\,\pi^2 C(j) \int_{S^1} \mathrm{d}t\ \Big\{\dot{\psi}(t;k)^2 - \big(1-\psi(t;k)^2\big)^2\Big\} \qquad \text{for} \quad k = \bar{k}\,, \tag{6.33}$$

where $\psi(t;k)$ is periodic and given in (6.20). This integral is finite and independent of t_0 but cannot be written down analytically. Its numerical value is about 41% of (6.31). Finally, the action functional on the Abelian solution (6.25) vanishes,

$$S_{\text{Abelian}} = 0\,, \tag{6.34}$$

because the integral of the electric and magnetic energy densities are finite and equal.

Boundary values of the Yang-Mills solutions. Since anti-de Sitter space has a boundary, it is of interest to note the value our solutions take there. The infinite-action field components (6.13) and (6.15) as well as the finite-action fields (6.23b), (6.24b) and (6.25b) all carry the conformal factor $\cosh^{-2}z = \cos^2\chi$, which vanishes at the boundary $z \to \pm\infty$ or $\chi = \pm\frac{\pi}{2}$. Therefore, our solutions live in the subspace of gauge fields decaying to zero at the AdS_4 boundary.

7 Self-dual Yang-Mills fields on anti-de Sitter space AdS_4

Euclidean AdS_4. Finally we discuss instantons in anti-de Sitter space AdS_4. The Euclidean continuation of the AdS_3 metric (C.2) is obtained by substituting $t = \mathrm{i}\tau$, which turns the z=const slices to 3-dimensional hyperbolic spaces H^3. The metric on AdS_4 transforms to a cosh-cone metric on the hyperbolic space H^4. This form of metric on H^4 is not convenient for our study of instantons since the natural boundary of H^4 is the 3-sphere $S^3 = \partial H^4$. However, there exist various other choices of coordinates and metrics on AdS_4 (see e.g. [20]), such as

$$\mathrm{d}s^2 = R^2\big(-\cosh^2\!\rho\,\mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2\!\rho\,\mathrm{d}\Omega_2^2\big)\,, \tag{7.1a}$$

$$\mathrm{d}s^2 = R^2\big(-\mathrm{d}t^2 + \sin^2\!t\,(\mathrm{d}\rho^2 + \sinh^2\!\rho\,\mathrm{d}\Omega_2^2)\big)\,, \tag{7.1b}$$

$$\mathrm{d}s^2 = R^2\big(\mathrm{d}\rho^2 + \sinh^2\!\rho\,(-\mathrm{d}t^2 + \cosh^2\!t\,\mathrm{d}\Omega_2^2)\big)\,. \tag{7.1c}$$

Choosing $t = \mathrm{i}(\chi-\frac{\pi}{2})$ in (7.1c) one obtains for H^4 a sinh-cone metric over S^3,

$$\mathrm{d}s^2 = R^2\big(\mathrm{d}\rho^2 + \sinh^2\!\rho\,(\mathrm{d}\chi^2 + \sin^2\!\chi\,\mathrm{d}\Omega_2^2)\big) = R^2(\mathrm{d}\rho^2 + \sinh^2\!\rho\,\delta_{ab}\,e^a e^b)\,, \tag{7.2}$$

which is convenient for analyzing gauge instantons on AdS_4.[8] Here, e^a are the left-invariant one-forms on S^3 satisfying (2.4) and discussed in detail in appendix B. We remark that, due to the range $\rho > 0$ the metric (7.2) describes only one sheet of the two-sheeted hyperboloid in $\mathbb{R}^{4,1}$ as a complete model of Euclideanized AdS_4. Furthermore, we must eventually fix some boundary conditions for the gauge fields, in order to investigate stability, for instance. Here, we take the attitude to postpone this discussion and first learn about classical solutions for any kind of boundary condition.

Cylinder metric. In stereographic coordinates x^i, $i = 1, \ldots, 4$, the H^4 metric reads

$$\mathrm{d}s^2 = \frac{4\,R^4}{(r^2-R^2)^2}\,\delta_{ij}\,\mathrm{d}x^i\mathrm{d}x^j \quad \text{for} \quad r^2 = \delta_{ij}\,x^i x^j \;<\; R^2\,, \tag{7.3}$$

which resembles the metric (4.5) on S^4. The forms (7.3) and (7.2) are related by the coordinate transformation

$$r = R\,\tanh\frac{\rho}{2} \qquad \text{with} \quad r \in [0,R) \quad \Leftrightarrow \quad \rho \in [0,\infty)\,. \tag{7.4}$$

[8] Also convenient [21] is the Euclidean continuation of (7.1a). Instantons in AdS_4 and $\widetilde{\mathrm{AdS}}_4$ with this metric in the form (5.16) will be considered at the end of this section.

Further, the metric (7.2) is conformally equivalent to the metric (4.10) on the Euclidean cylinder,

$$\mathrm{d}s^2 = R^2(\mathrm{d}\rho^2 + \sinh^2\!\rho\,\mathrm{d}\Omega_3^2) = \frac{R^2}{\sinh^2\!T}(\mathrm{d}T^2 + \mathrm{d}\Omega_3^2) = \frac{R^2}{\sinh^2\!T}\,\mathrm{d}s^2_{\mathrm{cyl}}\,, \tag{7.5}$$

where

$$T = \log\tanh\frac{\rho}{2} \qquad\Longleftrightarrow\qquad \tanh\frac{\rho}{2} = \mathrm{e}^T \qquad\Longleftrightarrow\qquad \sinh\rho = \frac{1}{\sinh T}\,. \tag{7.6}$$

BPST-type quasi-instanton. The Yang-Mills self-duality equations (4.9) are valid on any four-manifold. For the metric (7.5) on H^4 they are reduced to the equations on the cylinder $\mathbb{R}\times S^3$ with the metric (4.10) and become the generalized Nahm equations (4.12) for three matrices X_a. Therefore, we can copy the kink solution presented in section 4,

$$\mathcal{A} = X_a\,e^a\,, \qquad X_a = \frac{1}{2}(1+\psi)\,I_a\,, \qquad \psi(T) = \tanh(T{-}T_0)\,, \tag{7.7}$$

where the I_a are defined by (3.5) and T_0 is a real parameter. Thus we see that up to this moment the analysis of the self-dual Yang-Mills equations is the same on $\mathbb{R}^4$ (as a metric-cone over S^3), on S^4 (as a sine-cone over S^3), or on H^4 (as a sinh-cone over S^3). The differences appear only in the range of T and in the role of moduli parameter T_0.

First, for the cylinder $\mathbb{R}{\times}S^3$ we have $T\in(-\infty,\infty)$, yielding

$$\mathcal{A}(T{=}{-}\infty) = 0 \qquad\text{and}\qquad \mathcal{A}(T{=}{+}\infty) = e^a I_a = g^{-1}\mathrm{d}\,g\,, \tag{7.8}$$

where $g(\chi,\theta,\phi) : S^3 \to \mathrm{SU}(2)$ is a smooth map of degree (winding number) one. Thus, $\mathcal{A}(T)$ describes a transition from the trivial vacuum (sector of topological charge $q{=}0$) to a nontrivial vacuum (sector $q{=}1$). Second, for the sphere S^4 one takes $T\in[-\infty,\infty]$, corresponding to $\varphi\in[0,\pi]$. Hence, the self-dual solution again has topological charge $q{=}1$ and extends the one from $\mathbb{R}^4$ to S^4. In more detail, for the gauge field from (7.7) we get

$$\mathcal{F} = -\dot{X}_a\,e^a{\wedge}e^4 + \frac{1}{2}\left(-2\varepsilon^a_{bc}X_a + [X_b,X_c]\right)e^b{\wedge}e^c = \frac{1}{4\cosh^2(T{-}T_0)}\,\eta^a_{ij}\,e^i{\wedge}e^j I_a\,, \tag{7.9}$$

where $e^4 := \mathrm{d}T$. It follows that

$$q := -\frac{1}{16\pi^2 C(j)}\int_{\mathbb{R}\times S^3}\mathrm{tr}(\mathcal{F}{\wedge}\mathcal{F}) = 1\,. \tag{7.10}$$

This integral depends neither on the metric nor on T_0 and is the same for $\mathbb{R}^4$, $\mathbb{R}{\times}S^3$ and S^4.

Third, turning to hyperbolic space H^4, we see that

$$r\in[0,R) \qquad\Longleftrightarrow\qquad \rho\in[0,\infty) \qquad\Longleftrightarrow\qquad T\in[-\infty,0)\,. \tag{7.11}$$

This means that our solution (7.7) and (7.9) is defined only on the *half* line $\mathbb{R}_-$ and describes a transition

$$\begin{aligned} &\text{from} \qquad \mathcal{A}(T{=}{-}\infty) = 0 \qquad \text{to} \qquad \mathcal{A}(T{=}0) = \frac{1}{2}(1{-}\tanh T_0)\,e^a I_a\,, \\ &\text{i.e.} \qquad \mathcal{F}(T{=}{-}\infty) = 0 \qquad \text{to} \qquad \mathcal{F}(T{=}0) = \frac{1}{4\cosh^2\!T_0}\,\eta^a_{ij}\,e^i{\wedge}e^j I_a\,, \end{aligned} \tag{7.12}$$

connecting the trivial vacuum with an instanton section of size $\Lambda = \mathrm{e}^{T_0} R$, as discussed in [22]. Its quasi-topological charge depends on the moduli parameter T_0,

$$q(T_0) = -\frac{1}{16\pi^2 C(j)} \int_{\mathbb{R}_-\times S^3} \mathrm{tr}(\mathcal{F}\wedge\mathcal{F}) = \frac{3\,\mathrm{e}^{-T_0} + \mathrm{e}^{-3T_0}}{8\,\cosh^3 T_0} \tag{7.13}$$

ranging from $q{=}0$ for $T_0{\to}\infty$ to $q{=}1$ for $T_0{\to}{-}\infty$. For $T_0{=}0$ the boundary configuration sits in the middle of the kink, and so the solution (7.7) and (7.9) on $\mathbb{R}_-$ corresponds to a meron [23] (a singular non-self-dual Yang-Mills solution) which has topological charge $q = \frac{1}{2}$ in agreement with (7.13). We also see that for self-dual configurations on $\mathbb{R}_-{\times}S^3$ (and hence on H^4) the action functional $S = 4\pi^2 C(j)\,|q|$ decreases monotonically with T_0.

Geometric quasi-instanton. When studing instantons on AdS_4, one more possibility opens up. The simple flow equation (4.19) has, besides the kink in (7.7), also the singular solution

$$\psi(T) = \coth(T{-}T_0)\,. \tag{7.14}$$

It must be discarded on $\mathbb{R}^4$ and on S^4 due to the pole at $T = T_0$. However, for $T_0 > 0$ there is no singularity on the domain $[-\infty, 0) \ni T$ relevant for H^4. Substituting (7.14) into (7.7), we obtain the self-dual solution

$$\mathcal{A} = \frac{1}{2}\big(1 + \coth(T{-}T_0)\big)\, e^a I_a \qquad \text{and} \qquad \mathcal{F} = \frac{1}{4\sinh^2(T{-}T_0)}\, \eta^a_{ij}\, e^i{\wedge}e^j\, I_a\,. \tag{7.15}$$

With (4.8) and (4.22) we get

$$\frac{1}{4\sinh^2(T{-}T_0)} = \frac{\Lambda^2\, r^2}{(r^2{-}\Lambda^2)^2}\,, \tag{7.16}$$

and (7.15) coincides with the self-dual Yang-Mills configuration on H^4 naturally appearing in the geometric construction of [24], for example.[9] The topological charge of (7.15) comes out as

$$q(T_0) = -\frac{1}{16\pi^2 C(j)} \int_{\mathbb{R}_-\times S^3} \mathrm{tr}(\mathcal{F}\wedge\mathcal{F}) = \frac{3\,\mathrm{e}^{-T_0} - \mathrm{e}^{-3T_0}}{8\,\sinh^3 T_0}\,, \tag{7.17}$$

where now $T_0 > 0$. Thus, just like (7.9), the solution (7.15) has finite action.

Instantons in $\widetilde{\mathbf{AdS}}_4$. For a fuller picture we take a look at self-dual solutions on the universal cover $\widetilde{\mathrm{AdS}}_4$. To this end we perform a Euclidean continuation of the metric (5.16) as proposed in [21],

$$\mathrm{d}s^2 = \frac{R^2}{\cos^2\chi}\big(\mathrm{d}T^2 + \mathrm{d}\chi^2 + \sin^2\chi\,\mathrm{d}\Omega_2^2\big) = \frac{R^2}{\cos^2\chi}\big(\mathrm{d}T^2 + \mathrm{d}\Omega^2_{3+}\big) \quad \text{with} \quad \chi \in [0, \tfrac{\pi}{2})\,, \tag{7.18}$$

[9] On $S^4{=}\mathrm{Sp}(2)/\mathrm{Sp}(1){\times}\mathrm{Sp}(1)$ the instanton is naturally described as the self-dual part of the Levi-Civita connection in the fibration $\mathrm{Sp}(2)/\mathrm{Sp}(1) \to S^4$. Analogously, (7.15) is the self-dual part of the Levi-Civita connection on $H^4{=}\mathrm{Sp}(1{,}1)/\mathrm{Sp}(1){\times}\mathrm{Sp}(1)$, which is a connection in the fibration $\mathrm{Sp}(1{,}1)/\mathrm{Sp}(1) \to H^4$. Here, Sp(1,1) is the non-compact subgroup of $\mathrm{Sp}(2){\otimes}\mathbb{C}$ preserving the indefinite metric diag(1,-1) on the quaternionic space $\mathbb{H}^2$.

where $T \in S^1$ for AdS_4 but $T \in \mathbb{R}$ for $\widetilde{\mathrm{AdS}}_4$. As before, S^3_+ denotes the upper hemisphere, with a volume of π^2. The conformal boundary of this metric corresponds to $\chi = \frac{\pi}{2}$ and has the topology of a Euclidean space $S^1 \times S^2$ or $\mathbb{R} \times S^2$, respectively.

As before, the self-duality equations reduce to equations on the cylinder, but over S^3_+ instead of S^3. After taking the canonical ansatz (4.11) and specializing to (4.13) and (4.18) we again obtain the flow equation (4.19). However, this equation admits no periodic solution, so we do not find BPS configurations on Euclideanized AdS_4 in this way. On the other hand, the Euclildean version of the universal cover $\widetilde{\mathrm{AdS}}_4$ relaxes the periodicity requirement. Therefore, on this space we can take the canonical kink solution (7.7) defined for $T \in \mathbb{R}$. Then the gauge field (7.9) has the unit topological charge (7.10). Thus, standard instantons are well defined on the Euclidean version of the universal covering $\widetilde{\mathrm{AdS}}_4$ of anti-de Sitter space.

Non-self-dual Yang-Mills solutions on Euclideanized AdS_4 can nevertheless be found. The trusted ansatz (4.11), (4.13) and (4.18) reduces the full Yang-Mills equations to

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}T^2} = -2\,\psi\,(1-\psi^2)\,, \tag{7.19}$$

whose periodic solutions in terms of Jacobi elliptic functions are described e.g. in [13]. Substituting back into the ansatz yields non-self-dual finite-action Yang-Mills configurations, which describe a sequence of instanton-anti-instanton pairs.

8 Conclusions

We have established the existence of solitonic classical pure Yang-Mills configurations with finite energy and action in four-dimensional de Sitter and anti-de Sitter spaces. No Higgs fields are required. On de Sitter space dS_4 described as spatial S^3 slices over real time, our Yang-Mills solutions are spatially homogeneous and decay exponentially for early and late times. Replacing S^3 with AdS_3 yields infinite-action configurations on AdS_4. However, on anti-de Sitter space AdS_4 parametrized as spatial H^3 slices over a temporal circle, we again constructed solutions having finite energy and action, which decay exponentially in the radial direction of the hyperbolic slices.[10] For the Euclideanized version of $\mathbb{R}^{3,1}$, dS_4 and AdS_4, our method reproduces the known BPST instantons and lifts them from $\mathbb{R}^4$ to S^4 and H^4, respectively, where in the latter case we get only 'half' the instanton.

Due to their finite action, the described gauge configurations should be relevant in a semiclassical analysis of the path integral for quantum Yang-Mills theory on dS_4 or AdS_4. Their existence indicates that the Yang-Mills vacuum structure may depend on the cosmological constant, and the question of their stability calls for a computation of the (one-loop) effective action around these field configurations. One might hope to employ the (anti-)de Sitter radius R as a regulator towards quantum Yang-Mills theory on Minkowski space.

The most symmetric solution has an elementary geometric dependence on de Sitter time or on anti-de Sitter radial distance, and its action in both cases takes the minimal

[10]In the construction we strongly employed the conformal equivalence of H^3 to a 3-hemisphere S^3_+, see (5.14)–(5.19).

value of $-3\pi^3$ (for the SU(2) adjoint representation with normalization (3.5)), independent the (anti-)de Sitter radius. We conjecture this to be a lower bound for Yang-Mills solutions on these backgrounds. It will be important to investigate the stability of our configurations for certain boundary conditions.

Our solutions derive from three simplifying ansätze. First, we restricted the gauge potential to an $su(2)$ subalgebra and made an SU(2)-equivariant ansatz, which turns the Yang-Mills equations into ordinary coupled differential equations for three matrices. Second, we took these matrices to be proportional to the SU(2) generators, which produces a 3-dimensional Newtonian dynamical system with tetrahedral (S_4) symmetry. Third, we focus on stable submanifolds in the parameter space, which enables us to find analytic solutions.

At each step, generalizations are possible. First, one may admit a larger gauge group and a more general ansatz for the matrices, which will lead to quiver gauge theories. Second, it is tempting to analyze the matrix dynamics directly, for the potential and superpotential

$$V = -\mathrm{tr}\left\{ 2\, X_a X_a - \varepsilon_{abc}\, X_a\, [X_b, X_c] + \frac{1}{2}\, [X_a, X_b]\, [X_a, X_b] \right\}, \tag{8.1}$$

$$U = -\mathrm{tr}\left\{ X_a X_a - \frac{1}{6}\, \varepsilon_{abc}\, X_a\, [X_b, X_c] \right\}, \tag{8.2}$$

respectively. And third, for a good understanding of the analog Newtonian system one should investigate also numerical solutions for its full 3-dimensional dynamics. We hope to address these issues in the near future.

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A Four-dimensional metrics used in this paper

Metrics on dS_4.

$\mathrm{d}s^2/R^2$	coordinates	range
$-\mathrm{d}\tau^2 + \cosh^2\!\tau\, \mathrm{d}\Omega_3^2$	$(\tau, \chi, \theta, \phi)$	$\tau \in \mathbb{R}$
$\cos^{-2}\!t\, \left(-\mathrm{d}t^2 + \mathrm{d}\Omega_3^2\right)$	(t, χ, θ, ϕ)	$t \in (-\frac{\pi}{2}, \frac{\pi}{2})$
$-(1-\rho^2)\, \mathrm{d}\sigma^2 + \frac{\mathrm{d}\rho^2}{1-\rho^2} + \rho^2 \mathrm{d}\Omega_2^2$	$(\sigma, \rho, \theta, \phi)$	$\sigma \in \mathbb{R},\ \rho \in [0, 1)$

Metrics on S^4.

$\mathrm{d}s^2/R^2$	coordinates	range
$\mathrm{d}\varphi^2 + \sin^2\!\varphi\, \mathrm{d}\Omega_3^2$	$(\varphi, \chi, \theta, \phi)$	$\varphi \in [0, \pi]$
$\frac{4\, R^2}{(r^2+R^2)^2}\, \left(\mathrm{d}r^2 + r^2\, \mathrm{d}\Omega_3^2\right)$	(r, χ, θ, ϕ)	$r \in \mathbb{R}_+$
$\cosh^{-2}\!T\, \left(\mathrm{d}T^2 + \mathrm{d}\Omega_3^2\right)$	(T, χ, θ, ϕ)	$T \in \mathbb{R}$

Metrics on AdS_4.

$\mathrm{d}s^2/R^2$	coordinates	range
$\mathrm{d}z^2+\cosh^2 z\,\mathrm{d}\Omega_{2,1}^2$	(z,t,ρ,ϕ)	$z\in\mathbb{R}$
$\cos^{-2}\chi\left(\mathrm{d}\chi^2+\mathrm{d}\Omega_{2,1}^2\right)$	(χ,t,ρ,ϕ)	$\chi\in(-\frac{\pi}{2},\frac{\pi}{2})$
$-\cosh^2\rho\,\mathrm{d}t^2+\mathrm{d}\rho^2+\sinh^2\rho\,\mathrm{d}\Omega_2^2$	(t,ρ,θ,ϕ)	$t\in[-\pi,\pi]$, $\rho\in\mathbb{R}_+$
$\cos^{-2}\chi\left(-\mathrm{d}t^2+\mathrm{d}\Omega_{3+}^2\right)$	(t,χ,θ,ϕ)	$t\in[-\pi,\pi]$
$-\mathrm{d}t^2+\sin^2 t\left(\mathrm{d}\rho^2+\sinh^2\rho\,\mathrm{d}\Omega_2^2\right)$	(t,ρ,θ,ϕ)	$t\in[-\pi,\pi]$, $\rho\in\mathbb{R}_+$
$\mathrm{d}\rho^2+\sinh^2\rho\left(-\mathrm{d}t^2+\cosh^2 t\,\mathrm{d}\Omega_2^2\right)$	(ρ,t,θ,ϕ)	$t\in\mathbb{R}$, $\rho\in\mathbb{R}_+$

Metrics on H^4.

$\mathrm{d}s^2/R^2$	coordinates	range
$\mathrm{d}\rho^2+\sinh^2\rho\,\mathrm{d}\Omega_3^2$	(ρ,χ,θ,ϕ)	$\rho\in\mathbb{R}_+$
$\frac{4R^2}{(r^2-R^2)^2}\left(\mathrm{d}r^2+r^2\,\mathrm{d}\Omega_3^2\right)$	(r,χ,θ,ϕ)	$r\in[0,R)$
$\sinh^{-2}T\left(\mathrm{d}T^2+\mathrm{d}\Omega_3^2\right)$	(T,χ,θ,ϕ)	$T\in\mathbb{R}_-$
$\cos^{-2}\chi\left(\mathrm{d}T^2+\mathrm{d}\Omega_3^2\right)$	(T,χ,θ,ϕ)	$\chi\in[0,\frac{\pi}{2})$

B Metrics on S^3

A standard embedding of S^3 into $\mathbb{R}^4$ is given by

$$\omega^1=\sin\chi\,\sin\theta\,\sin\phi\,,\quad \omega^2=\sin\chi\,\sin\theta\,\cos\phi\,,\quad \omega^3=\sin\chi\,\cos\theta\,,\quad \omega^4=\cos\chi\,,\tag{B.1}$$

where $0\le\chi,\theta\le\pi$ and $0\le\phi<2\pi$. It induces on S^3 the metric

$$\mathrm{d}\Omega_3^2=\mathrm{d}\chi^2+\sin^2\chi\,(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)\,.\tag{B.2}$$

For $0\le\chi<\frac{\pi}{2}$ this is the metric on the 3-ball S^3_+, for $\frac{\pi}{2}<\chi\le\pi$ it is the metric on the 3-ball S^3_-, and for $\chi=\frac{\pi}{2}$ we have the equatorial S^2, in the decomposition $S^3=S^3_+\cup S^2\cup S^3_-$. Employing (2.5) the corresponding one-forms $\{e^a\}$ read

$$\begin{aligned}
e^1&=\sin\theta\sin\phi\,\mathrm{d}\chi+\sin\chi\cos\chi\,(\tan\chi\cos\phi+\cos\theta\sin\phi)\,\mathrm{d}\theta+\sin^2\chi\sin\theta\,(\cot\chi\cos\phi-\cos\theta\sin\phi)\,\mathrm{d}\phi\,,\\
e^2&=\sin\theta\cos\phi\,\mathrm{d}\chi-\sin\chi\cos\chi\,(\tan\chi\sin\phi-\cos\theta\cos\phi)\,\mathrm{d}\theta-\sin^2\chi\sin\theta\,(\cot\chi\sin\phi+\cos\theta\cos\phi)\,\mathrm{d}\phi\,,\\
e^3&=\cos\theta\,\mathrm{d}\chi-\sin\chi\cos\chi\sin\theta\,\mathrm{d}\theta+\sin^2\chi\sin^2\theta\,\mathrm{d}\phi\,,
\end{aligned}\tag{B.3}$$

in terms of which the metric reads

$$\mathrm{d}\Omega_3^2=\left(e^1\right)^2+\left(e^2\right)^2+\left(e^3\right)^2.\tag{B.4}$$

A simpler expression for $\{e^a\}$ arises from the different embedding choice

$$\omega^1=\cos\chi\cos\frac{\theta}{2}\,,\quad \omega^2=-\sin\chi\cos\frac{\theta}{2}\,,\quad \omega^3=\cos(\phi-\chi)\sin\frac{\theta}{2}\,,\quad \omega^4=\sin(\phi-\chi)\sin\frac{\theta}{2}\,,\tag{B.5}$$

where the angles $0 \leq \theta \leq \pi$ and $0 \leq \phi, \chi \leq 2\pi$ differ from those used above (but are denoted the same). For (B.5) substituted in (2.5) one obtains

$$\begin{aligned} e^1 &= \frac{1}{2}\big(\sin(2\chi{-}\phi)\,\mathrm{d}\theta - \cos(2\chi{-}\phi)\sin\theta\,\mathrm{d}\phi\big)\,, \\ e^2 &= \frac{1}{2}\big(\cos(2\chi{-}\phi)\,\mathrm{d}\theta + \sin(2\chi{-}\phi)\sin\theta\,\mathrm{d}\phi\big)\,, \\ e^3 &= \frac{1}{2}\big(\mathrm{d}(2\chi{-}\phi) + \cos\theta\,\mathrm{d}\phi\big) = \mathrm{d}\chi - \frac{1}{2}(1{-}\cos\theta)\,\mathrm{d}\phi\,. \end{aligned} \tag{B.6}$$

Correspondingly, the induced metric on the unit 3-sphere reads

$$\mathrm{d}\Omega_3^2 = \delta_{ab}\, e^a e^b = \Big(\mathrm{d}\chi - \frac{1}{2}(1{-}\cos\theta)\,\mathrm{d}\phi\Big)^2 + \frac{1}{4}\big(\mathrm{d}\theta^2 + \sin^2\!\theta\,\mathrm{d}\phi^2\big)\,. \tag{B.7}$$

This metric is adapted to the Hopf fibration

$$\pi\ :\quad S^3 \xrightarrow{\mathrm{U}(1)} S^2 \tag{B.8}$$

with the one-monopole connection[11]

$$a_1 = -\frac{\mathrm{i}}{2}\,(1 - \cos\theta\,\mathrm{d}\phi) \quad\Rightarrow\quad f_1 = \mathrm{d}a_1 = -\frac{\mathrm{i}}{2}\,\sin\theta\,\mathrm{d}\theta{\wedge}\mathrm{d}\phi \quad \text{and} \quad \frac{\mathrm{i}}{2\pi}\int_{S^2} f_1 = 1 \tag{B.9}$$

entering the metric (B.7).

C Metrics on AdS_3

A standard embedding of AdS_3 into $\mathbb{R}^{2,2}$ is given by

$$\omega^1 = \sinh\rho\,\cos\phi\,,\quad \omega^2 = \sinh\rho\,\sin\phi\,,\quad \omega^3 = \cosh\rho\,\cos t\,,\quad \omega^4 = \cosh\rho\,\sin t\,, \tag{C.1}$$

where $-\pi \leq t < \pi$, $\rho \geq 0$ and $0 \leq \phi < 2\pi$. It induces on AdS_3 the metric

$$\mathrm{d}\Omega_{2,1}^2 = -\cosh^2\!\rho\,\mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2\!\rho\,\mathrm{d}\phi^2\,. \tag{C.2}$$

One can introduce an orthonormal basis $\{e^\alpha\}$ of left-invariant one-forms

$$\begin{aligned} e^0 &= -\cosh^2\!\rho\,\mathrm{d}t - \sinh^2\!\rho\,\mathrm{d}\phi\,, \\ e^1 &= \sin(t{-}\phi)\,\mathrm{d}\rho - \sinh\rho\,\cosh\rho\,\cos(t{-}\phi)\,\mathrm{d}(t{+}\phi)\,, \\ e^2 &= -\cos(t{-}\phi)\,\mathrm{d}\rho - \sinh\rho\,\cosh\rho\,\cos(t{-}\phi)\,\mathrm{d}(t{+}\phi)\,, \end{aligned} \tag{C.3}$$

in terms of which the metric reads

$$\mathrm{d}\Omega_{2,1}^2 = -\big(e^1\big)^2 + \big(e^2\big)^2 + \big(e^3\big)^2\,. \tag{C.4}$$

D Yang-Mills solutions on dS_4 in various coordinates

We have constructed pure SU(2) Yang-Mills solutions in dS_4 parametrized by classical double-well trajectories $\psi(t)$. It is remarkable that their action is finite. Their scale is set by the inverse de Sitter radius R^{-1}. Here we display these solutions in different coordinates on de Sitter space.

[11] This is the form on the patch of S^2 around $\theta{=}0$. Around $\theta{=}\pi$ one should take $a_1 = \frac{\mathrm{i}}{2}\,(1 + \cos\theta\,\mathrm{d}\phi)$.

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D.1 Yang-Mills configuration in closed slicing

The solutions found in [9] for the closed slicing and described in more detail in section 3 depend on a suitable function $\psi(\tau)$ and have the form

$$\mathcal{A} = \frac{1}{2}\,(1{+}\psi)\,e^a I_a \qquad \text{and} \qquad \mathcal{F} = \left(\frac{1}{2}\,\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\,\mathrm{d}\tau{\wedge}e^a - \frac{1}{4}\,(1{-}\psi^2)\,\varepsilon^a_{\ bc}\,e^b{\wedge}e^c\right) I_a\,, \tag{D.1}$$

with three SU(2) generators $\{I_a\}$ and left-invariant one-forms $\{e^a\}$ obeying

$$[I_b, I_c] = 2\,\varepsilon^a_{\ bc}\,I_a\,, \qquad \mathrm{tr}(I_a I_b) = -4\,C(j)\,\delta_{ab}\,, \qquad \mathrm{d}e^a + \varepsilon^a_{\ bc}\,e^b\wedge e^c = 0\,, \tag{D.2}$$

where $C(j) = \frac{1}{3}j(j{+}1)(2j{+}1)$ is the second-order Dynkin index of the spin-j representation.

For extracting the components of $\mathcal{A}$ and $\mathcal{F}$, it is convenient to define three matrices I_* via

$$e^a I_a =: \mathrm{d}\chi\, I_\chi + \mathrm{d}\theta\, I_\theta + \mathrm{d}\phi\, I_\phi\,, \tag{D.3}$$

from which it follows that

$$\frac{1}{2}\varepsilon^a_{\ bc}\,e^b{\wedge}e^c\,I_a = -\frac{1}{\sin\theta}\,\mathrm{d}\chi{\wedge}\mathrm{d}\theta\,I_\phi + \sin\theta\,\mathrm{d}\chi{\wedge}\mathrm{d}\phi\,I_\theta - \sin^2\!\chi\,\sin\theta\,\mathrm{d}\theta{\wedge}\mathrm{d}\phi\,I_\chi\,. \tag{D.4}$$

In the fundamental representation of SU(2), $I_a = -\mathrm{i}\,\sigma_a$ and $C(\frac{1}{2}) = \frac{1}{2}$, and so from

$$e^a I_a = -\mathrm{i}\begin{pmatrix} e^3 & e^1{-}\mathrm{i}e^2 \\ e^1{+}\mathrm{i}e^2 & -e^3 \end{pmatrix} \qquad \text{and} \qquad \frac{1}{2}\varepsilon^a_{\ bc}\,e^b{\wedge}e^c\,I_a = -\mathrm{i}\begin{pmatrix} e^{12} & e^{23}{-}\mathrm{i}e^{31} \\ e^{23}{+}\mathrm{i}e^{31} & -e^{12} \end{pmatrix} \tag{D.5}$$

we compute

$$\begin{aligned}
I_\chi &= -\mathrm{i}\begin{pmatrix} \cos\theta & -\mathrm{i}\sin\theta\,\mathrm{e}^{\mathrm{i}\phi} \\ \mathrm{i}\sin\theta\,\mathrm{e}^{-\mathrm{i}\phi} & -\cos\theta \end{pmatrix}, \\
I_\theta &= -\mathrm{i}\,\sin\chi\cos\chi\begin{pmatrix} -\sin\theta & (\tan\chi{-}\mathrm{i}\cos\theta)\mathrm{e}^{\mathrm{i}\phi} \\ (\tan\chi{+}\mathrm{i}\cos\theta)\mathrm{e}^{-\mathrm{i}\phi} & \sin\theta \end{pmatrix}, \\
I_\phi &= -\mathrm{i}\,\sin^2\!\chi\sin\theta\begin{pmatrix} \sin\theta & (\cot\chi{+}\mathrm{i}\cos\theta)\mathrm{e}^{\mathrm{i}\phi} \\ (\cot\chi{-}\mathrm{i}\cos\theta)\mathrm{e}^{-\mathrm{i}\phi} & -\sin\theta \end{pmatrix}.
\end{aligned} \tag{D.6}$$

In the adjoint representation of SU(2), $(I_a)_{ij} = -2\,\varepsilon_{aij}$ and $C(1) = 2$, hence from

$$e^a I_a = -2\begin{pmatrix} 0 & e^3 & -e^2 \\ -e^3 & 0 & e^1 \\ e^2 & -e^1 & 0 \end{pmatrix} \quad \text{and} \quad \frac{1}{2}\varepsilon^a_{\ bc}\,e^b{\wedge}e^c\,I_a = -2\begin{pmatrix} 0 & e^{12} & -e^{31} \\ -e^{12} & 0 & e^{23} \\ e^{31} & -e^{23} & 0 \end{pmatrix} \tag{D.7}$$

one finds

$$I_\chi = -2\begin{pmatrix} 0 & \cos\theta & -\sin\theta\cos\phi \\ -\cos\theta & 0 & \sin\theta\sin\phi \\ \sin\theta\cos\phi & -\sin\theta\sin\phi & 0 \end{pmatrix},$$

$$I_\theta = -2\sin\chi\cos\chi\begin{pmatrix} 0 & -\sin\theta & \tan\chi\sin\theta-\cos\theta\cos\phi \\ \sin\theta & 0 & \tan\chi\cos\theta+\cos\theta\sin\phi \\ -\tan\chi\sin\theta+\cos\theta\cos\phi & -\tan\chi\cos\theta-\cos\theta\sin\phi & 0 \end{pmatrix},$$

$$I_\phi = -2\sin^2\chi\sin\theta\begin{pmatrix} 0 & \sin\theta & \cot\chi\sin\phi+\cos\theta\cos\phi \\ -\sin\theta & 0 & \cot\chi\cos\phi-\cos\theta\sin\phi \\ -\cot\chi\sin\phi-\cos\theta\cos\phi & -\cot\chi\cos\phi+\cos\theta\sin\phi & 0 \end{pmatrix}. \tag{D.8}$$

From these expressions, it is straightforward to write down the components of $\mathcal{A}$ and $\mathcal{F}$ on the 3-sphere, namely $\mathcal{A}_\tau = 0$ and

$$\begin{aligned} &\mathcal{A}_\chi = \frac{1}{2}(1+\psi)\,I_\chi\,, && \mathcal{A}_\theta = \frac{1}{2}(1+\psi)\,I_\theta\,, && \mathcal{A}_\phi = \frac{1}{2}(1+\psi)\,I_\phi\,, \\ &\mathcal{F}_{\tau\chi} = \frac{1}{2}\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\,I_\chi\,, && \mathcal{F}_{\tau\theta} = \frac{1}{2}\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\,I_\theta\,, && \mathcal{F}_{\tau\phi} = \frac{1}{2}\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\,I_\phi\,, \\ &\mathcal{F}_{\chi\theta} = \frac{1}{2}(1-\psi^2)\,\frac{1}{\sin\theta}\,I_\phi\,, && \mathcal{F}_{\chi\phi} = -\frac{1}{2}(1-\psi^2)\,\sin\theta\,I_\theta\,, && \mathcal{F}_{\theta\phi} = \frac{1}{2}(1-\psi^2)\,\sin^2\chi\sin\theta\,I_\chi\,. \end{aligned} \tag{D.9}$$

The corresponding electric and magnetic field components are then read off as

$$\begin{aligned} &E_\chi = \mathcal{F}_{\tau\chi}\,, && E_\theta = \mathcal{F}_{\tau\theta}\,, && E_\phi = \mathcal{F}_{\tau\phi}\,, \\ &B_\chi = -\frac{1}{\sin^2\chi\sin\theta}\,\mathcal{F}_{\theta\phi}\,, && B_\theta = \frac{1}{\sin\theta}\,\mathcal{F}_{\chi\phi}\,, && B_\phi = -\sin\theta\,\mathcal{F}_{\chi\theta}\,, \end{aligned} \tag{D.10}$$

and we see that the geometry factors precisely cancel for the magnetic components, hence

$$\begin{aligned} &E_\chi = \frac{1}{2}\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\,I_\chi\,, && E_\theta = \frac{1}{2}\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\,I_\theta\,, && E_\phi = \frac{1}{2}\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\,I_\phi\,, \\ &B_\chi = -\frac{1}{2}(1-\psi^2)\,I_\chi\,, && B_\theta = -\frac{1}{2}(1-\psi^2)\,I_\theta\,, && B_\phi = -\frac{1}{2}(1-\psi^2)\,I_\phi\,. \end{aligned} \tag{D.11}$$

Inspecting the matrices we see that all components are completely regular.

To view our fields on de Sitter space, we introduce an orthonormal basis on dS_4,

$$\tilde{e}^0 := \mathrm{d}\tilde{\tau} = R\,\mathrm{d}\tau \quad \text{and} \quad \tilde{e}^a := R\,\cosh\tau\,e^a\,, \tag{D.12}$$

and expand

$$\mathcal{A} = \widetilde{\mathcal{A}}_a\,\tilde{e}^a \qquad \text{and} \qquad \mathcal{F} = \widetilde{\mathcal{F}}_{0a}\,\tilde{e}^0\wedge\tilde{e}^a + \frac{1}{2}\widetilde{\mathcal{F}}_{bc}\,\tilde{e}^b\wedge\tilde{e}^c \tag{D.13}$$

so that

$$\mathcal{A}_a = R\cosh\tau\,\widetilde{\mathcal{A}}_a\,, \qquad \mathcal{F}_{bc} = R^2\cosh^2\tau\,\widetilde{\mathcal{F}}_{bc}\,, \qquad \mathcal{F}_{\tau a} = R^2\cosh\tau\,\widetilde{\mathcal{F}}_{0a}\,. \tag{D.14}$$

Therefore, the electric and magnetic field components in closed-slicing coordinates are

$$\begin{aligned} &\widetilde{E}_\chi = \frac{1}{2}\frac{1}{R^2\cosh\tau}\frac{\mathrm{d}\psi}{\mathrm{d}\tau}I_\chi\,, && \widetilde{E}_\theta = \frac{1}{2}\frac{1}{R^2\cosh\tau}\frac{\mathrm{d}\psi}{\mathrm{d}\tau}I_\theta\,, && \widetilde{E}_\phi = \frac{1}{2}\frac{1}{R^2\cosh\tau}\frac{\mathrm{d}\psi}{\mathrm{d}\tau}I_\phi\,, \\ &\widetilde{B}_\chi = -\frac{1}{2}\frac{1-\psi^2}{R^2\cosh^2\tau}I_\chi\,, && \widetilde{B}_\theta = -\frac{1}{2}\frac{1-\psi^2}{R^2\cosh^2\tau}I_\theta\,, && \widetilde{B}_\phi = -\frac{1}{2}\frac{1-\psi^2}{R^2\cosh^2\tau}I_\phi\,. \end{aligned} \tag{D.15}$$

For our orthonormal frame, the electric and magnetic energy densities become

$$\begin{aligned}\tilde\rho_e &= -\frac14\,\mathrm{tr}\widetilde E_a\widetilde E_a = -\frac14\,\mathrm{tr}\Big\{E_\chi^2+\frac{1}{\sin^2\!\chi}E_\theta^2+\frac{1}{\sin^2\!\theta\,\sin^2\!\theta}E_\phi^2\Big\} = \frac{3\,C(j)}{4\,R^4\cosh^2\!\tau}\Big(\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\Big)^2\,,\\ \tilde\rho_m &= -\frac14\,\mathrm{tr}\widetilde B_a\widetilde B_a = -\frac14\,\mathrm{tr}\Big\{B_\chi^2+\frac{1}{\sin^2\!\chi}B_\theta^2+\frac{1}{\sin^2\!\theta\,\sin^2\!\theta}B_\phi^2\Big\} = \frac{3\,C(j)}{4\,R^4\cosh^4\!\tau}\big(1-\psi^2\big)^2\,.\end{aligned} \tag{D.16}$$

The de Sitter energy of the Yang-Mills configuration then turns out to be ($\dot\psi=\cosh\tau\,\frac{\mathrm{d}\psi}{\mathrm{d}\tau}$)

$$\mathcal{E}_{\tilde\tau} = \int_{S^3_R}\tilde e^1\wedge\tilde e^2\wedge\tilde e^3\;(\tilde\rho_e+\tilde\rho_m) = \frac34\,C(j)\,\mathrm{vol}(S^3_1)\,\frac{\dot\psi^2+(1-\psi^2)^2}{R\,\cosh\tau} = \frac32\pi^2\,C(j)\,\frac{1}{R\,\cosh\tau}\,. \tag{D.17}$$

D.2 Yang-Mills configuration in Hopf coordinates

The Hopf coordinates on S^3 described in (B.5)–(B.7) allow for a somewhat simpler form of the matrices I_* in the decomposition (D.3) by exploiting (B.6). We abbreviate $\tilde\chi:=2\chi-\phi$.

In the fundamental SU(2) representation, the potential and the field strength are given by

$$\mathcal{A} = -\frac{\mathrm{i}}{4}\,(1+\psi)\begin{pmatrix}\mathrm{d}\tilde\chi+\cos\theta\,\mathrm{d}\phi & \mathrm{e}^{\mathrm{i}\tilde\chi}(-\mathrm{i}\mathrm{d}\theta-\sin\theta\,\mathrm{d}\phi)\\ \mathrm{e}^{-\mathrm{i}\tilde\chi}(\mathrm{i}\mathrm{d}\theta-\sin\theta\,\mathrm{d}\phi) & -\mathrm{d}\tilde\chi-\cos\theta\,\mathrm{d}\phi\end{pmatrix} \qquad \text{and} \tag{D.18}$$

$$\begin{aligned}\mathcal{F} = &-\frac{\mathrm{i}}{4}\dot\psi\,\mathrm{d}\tau\wedge\begin{pmatrix}\mathrm{d}\tilde\chi+\cos\theta\,\mathrm{d}\phi & \mathrm{e}^{\mathrm{i}\tilde\chi}(-\mathrm{i}\mathrm{d}\theta-\sin\theta\,\mathrm{d}\phi)\\ \mathrm{e}^{-\mathrm{i}\tilde\chi}(\mathrm{i}\mathrm{d}\theta-\sin\theta\,\mathrm{d}\phi) & -\mathrm{d}\tilde\chi-\cos\theta\,\mathrm{d}\phi\end{pmatrix}+ \\ &+\frac{\mathrm{i}}{4}(1-\psi^2)\begin{pmatrix}\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi & \mathrm{e}^{\mathrm{i}\tilde\chi}(\mathrm{d}\theta\wedge\mathrm{d}\tilde\chi-\mathrm{i}\sin\theta\,\mathrm{d}\phi\wedge\mathrm{d}\tilde\chi+\cos\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi)\\ \mathrm{e}^{-\mathrm{i}\tilde\chi}(\mathrm{d}\theta\wedge\mathrm{d}\tilde\chi+\mathrm{i}\sin\theta\,\mathrm{d}\phi\wedge\mathrm{d}\tilde\chi+\cos\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi) & -\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi\end{pmatrix}.\end{aligned} \tag{D.19}$$

In the adjoint representation, we arrive at

$$\mathcal{A} = \frac12(1+\psi)\begin{pmatrix}0 & -\mathrm{d}\tilde\chi-\cos\theta\,\mathrm{d}\phi & \cos\tilde\chi\,\mathrm{d}\theta+\sin\tilde\chi\sin\theta\,\mathrm{d}\phi\\ \mathrm{d}\tilde\chi+\cos\theta\,\mathrm{d}\phi & 0 & -\sin\tilde\chi\,\mathrm{d}\theta+\cos\tilde\chi\sin\theta\,\mathrm{d}\phi\\ -\cos\tilde\chi\,\mathrm{d}\theta-\sin\tilde\chi\sin\theta\,\mathrm{d}\phi & \sin\tilde\chi\,\mathrm{d}\theta-\cos\tilde\chi\sin\theta\,\mathrm{d}\phi & 0\end{pmatrix}. \tag{D.20}$$

The expression for the field strength is straightforward but too lengthy to write down here.

D.3 Yang-Mills configuration in static slicing

With a common 2-sphere parametrization, the relation between the closed and the static slice is given by just two relations, e.g.

$$\sinh\tau = \sqrt{1-\rho^2}\,\sinh t = \cos\alpha\,\sinh t\,, \tag{D.21}$$

$$\sin\chi\,\cosh\tau = \rho = \sin\alpha\,, \tag{D.22}$$

from which one derives other relations, such as

$$\cos\chi\,\cosh\tau = \cos\alpha\,\cosh t\,. \tag{D.23}$$

A frequently used combination is

$$\Delta^2 := \cosh^2\tau = 1+\cos^2\alpha\,\sinh^2 t = \cosh^2 t - \sin^2\alpha\,\sinh^2 t = \sin^2\alpha + \cos^2\alpha\,\cosh^2 t\,. \qquad \text{(D.24)}$$

We may express the closed-slicing coordinates in terms of the static-slicing ones,

$$\tau = \operatorname{arsinh}(\cos\alpha\,\sinh t) \quad \text{and} \quad \chi = \arcsin(\sin\alpha\,\Delta^{-1})\,, \qquad \text{(D.25)}$$

with $\Delta = \Delta(\alpha, t)$. From this it is straightforward to derive

$$\begin{aligned} \frac{\partial\tau}{\partial t} &= \frac{\cos\alpha\,\cosh t}{\Delta}\,, & \frac{\partial\tau}{\partial\alpha} &= -\frac{\sin\alpha\,\sinh t}{\Delta}\,, \\ \frac{\partial\chi}{\partial t} &= -\frac{\sin\alpha\,\cos\alpha\,\sinh t}{\Delta^2}\,, & \frac{\partial\chi}{\partial\alpha} &= \frac{\cosh t}{\Delta^2}\,, \end{aligned} \qquad \text{(D.26)}$$

which provides the Jacobian for the change of 'closed' to 'static' variables.

In order to evaluate the components of the gauge potential and the field strength in the static coordinates (x^I), we transform them from the closed coordinates (x^i) according to

$$\mathcal{A}_I = \frac{\partial x^i}{\partial x^I}\,\mathcal{A}_i \quad \text{and} \quad \mathcal{F}_{IJ} = \frac{\partial x^i}{\partial x^I}\frac{\partial x^j}{\partial x^J}\,\mathcal{F}_{ij} \qquad \text{(D.27)}$$

and re-express the arguments $x^i = x^i(x^I)$, e.g.

$$\sin\chi \;\Rightarrow\; \Delta^{-1}\sin\alpha \quad \text{and} \quad \cos\chi \;\Rightarrow\; \Delta^{-1}\cos\alpha\,\cosh t\,. \qquad \text{(D.28)}$$

Since the S^2 coordinates θ and ϕ are common to both systems and we employ the $\mathcal{A}_\tau = 0$ gauge, we remain with

$$\mathcal{A}_t = \frac{\partial\chi}{\partial t}\,\mathcal{A}_\chi \quad \text{and} \quad \mathcal{A}_\rho = \frac{1}{\cos\alpha}\frac{\partial\chi}{\partial\alpha}\,\mathcal{A}_\chi \qquad \text{(D.29)}$$

and, since the determinant of the Jacobian equals Δ^{-1},

$$\begin{aligned} &\mathcal{F}_{t\rho} = \frac{1}{\Delta}\,\mathcal{F}_{\tau\chi}\,, \quad \mathcal{F}_{t\theta} = \frac{\partial\tau}{\partial t}\,\mathcal{F}_{\tau\theta} + \frac{\partial\chi}{\partial t}\,\mathcal{F}_{\chi\theta}\,, && \mathcal{F}_{t\phi} = \frac{\partial\tau}{\partial t}\,\mathcal{F}_{\tau\phi} + \frac{\partial\chi}{\partial t}\,\mathcal{F}_{\chi\phi}\,, \\ &\mathcal{F}_{\rho\theta} = \frac{1}{\cos\alpha}\Big(\frac{\partial\tau}{\partial\alpha}\,\mathcal{F}_{\tau\theta} + \frac{\partial\chi}{\partial\alpha}\,\mathcal{F}_{\chi\theta}\Big)\,, && \mathcal{F}_{\rho\phi} = \frac{1}{\cos\alpha}\Big(\frac{\partial\tau}{\partial\alpha}\,\mathcal{F}_{\tau\phi} + \frac{\partial\chi}{\partial\alpha}\,\mathcal{F}_{\chi\phi}\Big)\,. \end{aligned} \qquad \text{(D.30)}$$

In these coordinates electric and magnetic field components are defined as

$$\begin{aligned} E_\rho &= \mathcal{F}_{t\rho}\,, & E_\theta &= \mathcal{F}_{t\theta}\,, & E_\phi &= \mathcal{F}_{t\phi}\,, \\ B_\rho &= -\frac{1}{\rho^2\sin\theta}\,\mathcal{F}_{\theta\phi}\,, & B_\theta &= \frac{\cos^2\alpha}{\sin\theta}\,\mathcal{F}_{\rho\phi}\,, & B_\phi &= -\cos\alpha\sin\theta\,\mathcal{F}_{\rho\theta}\,. \end{aligned} \qquad \text{(D.31)}$$

Passing to the dimensional (tilded) coordinates multiplies these relations with a factor of

$$R^{-2}\cosh^{-2}\tau = R^{-2}\Delta^{-2}\,. \qquad \text{(D.32)}$$

The radial components expressed in terms of the static coordinates take a reasonably simple form,

$$\widetilde{E}_r = \frac{1}{R\,\Delta}\,\widetilde{\mathcal{F}}_{\tau\chi} = \frac{1}{2}\,\frac{1}{R^3\Delta^3}\,\dot{\psi}^2 I_\chi \quad \text{and} \quad \widetilde{B}_r = -\frac{1}{R\,\rho^2\sin\theta}\,\widetilde{\mathcal{F}}_{\theta\phi} = -\frac{1}{2}\,\frac{1}{R^3\Delta^4}(1-\psi^2)I_\chi\,. \qquad \text{(D.33)}$$

It will be interesting to physically interpret the static field components, in particular for the limits $\alpha \to 0$ ('center' of the configuration) and $\alpha \to 1$ (cosmological horizon).

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